Quantum Amplitudes in Black-Hole Evaporation: Spins 1 and 2

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Abstract

Quantum amplitudes for $s = 1$ Maxwell fields and for $s = 2$ linearised gravitational-wave perturbations of a spherically-symmetric Einstein/massless scalar background, describing gravitational collapse to a black hole, are treated by analogy with the previous treatment of $s = 0$ scalar-field perturbations of gravitational collapse at late times. Both the spin-1 and the spin-2 perturbations split into parts with odd and even parity. Their detailed angular behaviour is analysed, as well as their behaviour under infinitesimal coordinate transformations and their linearised field equations. In general, we work in the Regge-Wheeler gauge, except that, at a certain point, it becomes necessary to make a gauge transformation to an asymptotically-flat gauge, such that the metric perturbations have the expected fall-off behaviour at large radii. In both the $s = 1$ and $s = 2$ cases, we isolate suitable 'coordinate' variables which can be taken as boundary data on a final space-like hypersurface Σ_F. (For simplicity of exposition, we take the data on the initial surface Σ_I to be exactly spherically-symmetric.) The (large) Lorentzian proper-time interval between Σ_I and Σ_F, measured at spatial infinity, is denoted by T. We then consider the classical boundary-value problem and calculate the second-variation classical Lorentzian action $S^{(2)}_{\rm class}$, on the assumption that the time interval T has been rotated into the complex: $T \to |T| \exp(-i\theta)$, for $0 < \theta \leq \pi/2$. This complexified classical boundary-value problem is expected to be well-posed, in contrast to the boundary-value problem in the Lorentzian-signature case $(\theta = 0)$, which is badly posed, since it refers to hyperbolic or wave-like field equations. Following Feynman, we recover the Lorentzian quantum amplitude by taking the limit as $\theta \to 0_+$ of the semi-classical amplitude $\exp\left(iS^{(2)}_{\rm class}\right)$. The boundary data for $s = 1$ involve the (Maxwell) magnetic field, while the data for $s = 2$ involve the magnetic part of the Weyl curvature tensor. These relations are also investigated, using 2-component spinor language, in terms of the Maxwell field strength $\phi_{AB} = \phi_{(AB)}$ and the Weyl spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$. The magnetic boundary conditions are related to each other and to the natural $s = \frac{1}{2}$ boundary conditions by supersymmetry.

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1 Introduction

This paper describes part of a project concerned with the calculation of quantum amplitudes (not just probabilities) associated with quantum fields, including gravity itself, in the case that strong gravitational fields may be present. The most obvious example – the original motivation for this work – concerns quantum radiation associated with gravitational collapse to a black hole [1-11]. But the framework adopted here is more general, and certainly does not depend on whether there is a classical Lorentzian-signature collapse to a black hole. It includes the case of local collapse which is not sufficient to lead to (Lorentzian) curvature singularities, and also quantum processes in cosmology, where, for example, anisotropies in the Cosmic Microwave Background Radiation (CMBR) can be computed, and depend crucially on the underlying Lagrangian for gravity and matter [12].

To exemplify the underlying ideas, we consider the case of local collapse (whether or not to a black hole). Thus, the gravitational field is taken to be asymptotically flat. For simplicity, consider Einstein gravity coupled minimally to a massless scalar field ϕ. In classical gravitation, we are used to describing this by means of a Cauchy problem, giving evolution to the future (say) of an initial space-like hypersurface $\mathcal{S}$, which extends to spatial infinity. We write $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ for the components of the 4-dimensional metric, and then denote by $h_{ij} = g_{ij}$ $(i, j = 1, 2, 3)$ the components of the intrinsic spatial metric on $\mathcal{S}$ in the case that $\mathcal{S}$ is given by the condition $x^0 = \mathrm{const}$. Cauchy data would, loosely speaking, consist of h_{ij}, ϕ and their corresponding normal derivatives on $\mathcal{S}$. By contrast, in quantum theory, one typically asks for the amplitude to go from an initial configuration such as $(h_{ij}, \phi)_I$ on an initial hypersurface Σ_I, to a final configuration $(h_{ij}, \phi)_F$ on a final hypersurface Σ_F. The problem of finding the quantum amplitude should (naively) be completely posed, once one has also specified the (Lorentzian) proper-time interval between the surfaces Σ_I and Σ_F, as measured near spatial infinity.

Much of the 'non-intuitive' nature of quantum mechanics can be traced to the 'boundary-value' nature of such a quantum amplitude [13], as compared with the familiar classical initial-value problem. A crucial mathematical aspect of this difference, responsible for a good part of the 'non-intuition', is that the *classical* version of the problem of calculating a quantum amplitude, as posed above, would involve solving the classical field equations (typically hyperbolic), subject to the given boundary data $(h_{ij}, \phi)_{I,F}$ on the hypersurfaces Σ_I, Σ_F, separated near spatial infinity by a Lorentzian time interval T. As is well known, a boundary-value problem for a hyperbolic equation is typically not well posed. For typical boundary data, a classical solution will not exist 14,15]; or, if it does exist, it will be non-unique. The straightforward cure for this ill, due to Feynman [13], is of course to rotate the Lorentzian time-interval T into the complex: $T \to |T| \exp(-i\theta)$, with $0 < \theta \leq \pi/2$.

A simplified classical boundary-value example, showing this behaviour, is described in [16]. When this example is posed originally (and badly) in 2-dimensional Minkowski space-time, one considers a scalar field $\phi(t, x)$, obeying

the wave equation

$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2} \;=\; 0 \quad ; \qquad 0<t<T \;, \quad -\infty<x<+\infty \quad , \tag{1.1}$$

on the assumption that ϕ decays rapidly as $|x|\to\infty$. A simple choice of Dirichlet boundary data is to take

$$\phi(t=0\,,\,x) \;=\; 0 \quad , \qquad \phi(t=T\,,\,x) \;=\; \phi_1(x) \qquad . \tag{1.2}$$

The time-interval T at spatial infinity is then, as above, rotated into the lower complex half-plane:

$$T \;\to\; |T|\,\exp(-\,i\theta) \;; \qquad 0\;<\;\theta\leq\pi/2 \qquad . \tag{1.3}$$

For convenience, we define, for a given fixed θ $(0\;<\;\theta\leq\pi/2)$, the 'rotated-time' coordinate

$$y \;=\; t\,\exp(-\,i\theta) \quad . \tag{1.4}$$

In terms of the new coordinates $(y\,,x)$, the wave equation (1.1) reads

$$-\,e^{2i\theta}\,\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial x^2} \;=\; 0 \qquad , \tag{1.5}$$

and the boundary conditions (1.2) become

$$\phi(y=0\,,\,x) \;=\; 0 \quad , \qquad \phi\big(y=Te^{i\theta},\,x\big) \;=\; \phi_1(x) \qquad . \tag{1.6}$$

Here, the extreme case $\theta\;=\;\pi/2$ corresponds to the Riemannian (Euclidean) sector, and to a well-posed real elliptic Dirichlet boundary-value problem for the Laplace equation. The potential $\phi(y\,,x)$ is thus required to be a complex solution of Eqs.(1.5,6), which, for $0\;<\;\theta\;\leq\pi/2$, describe a *strongly elliptic* partial differential equation in the sense of [17]. The property of strong ellipticity guarantees existence and uniqueness in this linear example. As verified in [16], however, the 'classical solution' becomes singular in the Lorentzian limit $\theta\to 0_+$.

In our coupled non-linear gravitational/scalar-field example, the extreme case $\theta\;=\;\pi/2$ would correspond to a purely Euclidean time-interval $|T|$, and classically one would then be solving the field equations for Riemannian gravity with a scalar field ϕ. Since these field equations are 'elliptic *modulo* gauge' – see [17] – one would expect to have a well-posed classical boundary-value problem, with existence and uniqueness. The intermediate case $0\;<\;\theta\;<\;\pi/2$ requires the interval T and any classical solution to involve the complex numbers non-trivially. If the problem turns out to be strongly elliptic, up to gauge, then the complex case $0<\theta<\pi/2$ would again be expected to have the good existence and uniqueness properties of the real elliptic case.

In practice, in the black-hole evaporation problem or (say) in cosmological examples, one typically treats the case in which both the gravitational and the

scalar initial data are close to spherical symmetry. Hence, as a leading approximation, one begins by studying the spherically-symmetric Einstein/scalar system. This was treated in [18] for Lorentzian signature and is outlined in [19] for Riemannian signature. In the Riemannian case, the metric is taken (without loss of generality) in the form

$$ds^2 \quad = \quad e^b\, d\tau^2 \,+\, e^a\, dr^2 \,+\, r^2\,(d\theta^2 + \sin^2\theta\; d\phi^2) \quad , \tag{1.7}$$

where

$$b \;=\; b(\tau, r) \;, \qquad\qquad a \;=\; a(\tau, r) \qquad , \tag{1.8}$$

and the scalar field is taken of the form $\phi(\tau, r)$ [19]. The scalar field equation reads:

$$\ddot{\phi} \,+\, e^{b-a}\,\phi'' \,+\, \frac{1}{2}\left(\dot{a} - \dot{b}\right)\dot{\phi} \,+\, r^{-1}\,e^{b-a}\left(1 + e^a\right)\phi' \quad = \quad 0 \quad , \tag{1.9}$$

where $(\dot{\ })$ denotes $\partial(\ )/\partial\tau$ and $(\ )'$ denotes $\partial(\ )/\partial r$. Together with Eq.(1.9), a slightly redundant set of gravitational field equations is given by:

$$a' \quad = \; -\,4\pi r\left(e^{a-b}\,\dot{\phi}^2 - \phi'^2\right) + r^{-1}\left(1 - e^a\right) \quad , \tag{1.10}$$

$$b' \quad = \; -\,4\pi r\left(e^{a-b}\,\dot{\phi}^2 - \phi'^2\right) - r^{-1}\left(1 - e^a\right) \quad , \tag{1.11}$$

$$\dot{a} \quad = \quad 8\pi r\,\dot{\phi}\,\phi' \quad , \tag{1.12}$$

$$\ddot{a} + e^{b-a}\,b'' + \frac{1}{2}\left(\dot{a} - \dot{b}\right)\dot{a} - r^{-1}e^{b-a}\left(1 - e^a\right)\left(b' + 2r^{-1}\right) = 8\pi\left(\dot{\phi}^2 + e^{b-a}\,\phi'^2\right). \tag{1.13}$$

The metric and the classical field equations in Lorentzian signature [18] can be derived from the above by the formal replacement

$$t \quad = \quad \tau\, e^{-i\theta} \quad , \tag{1.14}$$

where $\theta = \pi/2$ is independent of 4-dimensional position. Similarly for complex metrics with suitable behaviour at infinity, with $0 < \theta \leq \pi/2$.

Even in the spherically-symmetric case, very little is known rigorously about existence and uniqueness for the Riemannian (or complex) boundary-value problem. For this case, numerical investigation of the weak-field Riemannian boundary-value problem was begun in [19], and has recently been extended towards the strong-field region [20]. For weak scalar boundary data, global quantities such as the mass M and Euclidean action I appear to scale quadratically, in accordance with analytic weak-field estimates [20]. In the limit of strong-field scalar boundary data, it may be that a typical pattern will emerge numerically for the general 'shape' of the classical Riemannian gravitational and scalar fields. In that case, it might be possible to find analytic approximations for the strong-field limit (quite different from those valid in the weak-field case), which could provide further analytical insight into the solutions of the coupled Riemannian Einstein/scalar boundary-value problem. In particular, it would be extremely valuable to have strong-field approximations which were valid into the complex region, with $0 < \theta < \pi/2$. One might conjecture that, as one approaches

the Lorentzian limit $\theta \rightarrow 0_+$, for very strong spherically-symmetric boundary data, the solutions (albeit complex) correspond to classical Einstein/scalar solutions which form a singularity, surrounded by a black hole.

In the case of quantum amplitudes for Lagrangians with Einstein gravity coupled to matter, one can consider *anisotropic* boundary data posed in the 'field language' of this paper, by taking (for the present Einstein/scalar case) non-spherically-symmetric boundary data $(h_{ij}\,,\phi)_{I,F}$. Then, at least in the asymptotically-flat case with time-interval T at spatial infinity, one is inevitably led to consider the complexified boundary-value problem, with $T \rightarrow |T|\,\exp(-\,i\theta)$, but with unchanged data $(h_{ij}\,,\phi)_{I,F}$ on the other boundaries. This corresponds (by a slight re-definition of θ) to the procedure adopted in our model 2-dimensional boundary-value problem of Eqs.(1.1-6). Even for fairly small θ, solution of this boundary-value problem is expected to smooth out variations or oscillations of the boundary data, when one moves into the interior by a few multiples of the relevant wavelength. If the problem is genuinely strongly elliptic, up to gauge, then one will be able to extend the classical solution analytically into the complex.

Strictly, in order that quantum amplitudes should be meaningful for any Einstein-gravity/matter Lagrangian under consideration, one should work only with theories invariant under local supersymmetry – that is, with supergravity models or supergravity coupled to supermatter [21,22]. Thus, for example, the bosonic Einstein/massless-scalar model above should be replaced by the simplest locally-supersymmetric theory which contains it [21]. This $N=1$ supergravity/supermatter model contains a *complex* scalar field ϕ, with a massless spin-$\frac{1}{2}$ partner; the graviton acquires a spin-$\frac{3}{2}$ gravitino partner. Generally, for 'Riemannian' boundary data, one expects that the resulting 'Euclidean' quantum amplitude has the semi-classical form

$$\mathrm{Amp} \quad \sim \quad (A_0 + \hbar A_1 + \hbar^2 A_2 + \ldots\,)\ \exp\bigl(-\,I_{\rm class}/\hbar\bigr) \quad , \tag{1.15}$$

asymptotically in the limit that $\bigl(I_{\rm class}/\hbar\bigr) \rightarrow 0$. Here, $I_{\rm class}$ is the classical 'Euclidean action' of a Riemannian solution of the coupled Einstein and bosonic-matter classical field equations, subject to suitable boundary conditions. In the complex régime of this paper, we shall use the expressions I and $-\,iS$ interchangeably, where S denotes the 'Lorentzian action'. For simplicity, we assume that there is a unique classical solution, up to gauge, coordinate and local supersymmetry transformations. But it is quite feasible, in certain theories and for certain boundary data, to have instead (say) a complex-conjugate pair of classical solutions [23]. The classical action $I_{\rm class}$ and loop terms $A_0\,, A_1\,, A_2\,, \ldots$ depend in principle on the boundary data. In the case of supermatter coupled to $N=1$ supergravity, each of $I_{\rm class}\,, A_0\,, A_1\,, A_2\,, \ldots$ will also obey differential constraints connected with the local coordinate and local supersymmetry invariance of the theory, and with any other local invariances such as gauge invariance (if appropriate) [15,24].

In particular, in the locally-supersymmetric case, the semi-classical expansion (1.15) may become extremely simple [15,25,26]. For example, for $N=1$

supergravity, for purely bosonic (Einstein) boundary data, one has [15]:

$$\text{Amp} \quad \sim \quad A_0 \exp\bigl(-I_{\text{class}}/\hbar\bigr) \quad , \tag{1.16}$$

where, further, the one-loop factor A_0 is in fact a constant. When the boundary data are allowed to include both bosonic and fermionic parts (suitably posed), one expects that a classical solution of the coupled bosonic/fermionic field equations will still exist. In this case, the expression I_{class} in Eq.(1.16) denotes the full classical action, including both bosonic and fermionic contributions. The fermionic contributions (naturally) also depend on the boundary data, and, as is standard in the holomorphic representation used here for fermions [27-29], live in a Grassmann algebra over the complex numbers. Related properties hold for $N = 1$ supergravity coupled to gauge-invariant supermatter [22,25]. Whether or not the supermatter is also invariant under a gauge group, there will be analogous consequences for the semi-classical expansion (1.15) of the quantum amplitude. In particular, one expects finite loop terms $A_0\,, A_1\,, A_2\,, \ldots$ [15,25,26], but typically not the maximal simplicity of the pure supergravity amplitude (1.16).

In the case (1.16) of pure supergravity, the classical action is all that is needed for the quantum computation. A corresponding situation arises with ultra-high-energy collisions, whether between black holes [30], in particle scattering [31], or in string theory [32].

To fix one's physical intuition, one can assume that, near the initial surface Σ_I , the gravitational and scalar fields are approximately spherically symmetric and vary extremely slowly with time, corresponding to diffuse bosonic matter near Σ_I . The final hypersurface Σ_F should preferably be taken at a sufficiently late time T that all the quantum radiation due to the evaporation of the black hole will by then have been emitted. The final data $(h_{ij}\,, \phi)_F$ for gravity and the scalar field, together, if a Maxwell field is included, with the spatial components $(A_i)_F$ of the vector potential, are taken to have small anisotropic parts – this corresponds, in 'particle language', to a choice of final particle state. The resulting 'weak-field' quantum amplitude, to be found below, can be described in terms of products of zero- or one-particle harmonic-oscillator eigenstates. One could, of course, also consider final data which deviate strongly from spherical symmetry. Their quantum amplitudes will still be roughly proportional to the (complex) quantity $\lim_{\theta\to 0_+} \exp(-I_{\text{class}})$. For the weak perturbations (above) away from spherical symmetry, I_{class} is nearly quadratic, but for strong perturbations, I_{class} will be very non-linear, and the probabilities of such final configurations will be microscopic.

In considering the classical boundary-value problem, even though spin-1 and fermionic fields may also be involved, we shall for simplicity first consider the fields $g_{\mu\nu}$ and ϕ . As above, we consider the Riemannian (or complex-rotated Riemannian) classical boundary-value problem, given boundary data $(h_{ij}\,, \phi)_I$ and $(h_{ij}\,, \phi)_F$, where the initial and final boundary hypersurfaces Σ_I and Σ_F are separated at spatial infinity by a complex (Riemannian-time) interval of the form $|T| \exp(-i\theta)$. Since all fields are to be regarded as perturbations of

a 'background' spherically-symmetric configuration, we are assuming that the classical solutions $(g_{\mu\nu}\,,\phi)$ of the coupled Einstein/scalar field equations may be decomposed into a 'background' spherically-symmetric part $(\gamma_{\mu\nu}\,,\Phi)$, together with a 'small' perturbative part. The linearised perturbative fields, whether spin-0 scalar [11,33,34], spin-1 or spin-2 (this paper), spin-$\frac{1}{2}$ [35] or spin-$\frac{3}{2}$ (in progress [36]), can be expanded out in the appropriate spin-weighted spherical harmonics [37-39].

Typically, the perturbative scalar-field configuration (say) $\phi^{(1)}_F$, given on the late-time surface Σ_F , will involve an enormous number of modes, both angular and radial, but with a minute coefficient for each mode. That is, the given $\phi^{(1)}_F$ may contain extremely detailed angular structure, and also be spread over a considerable radius from the centre of spherical symmetry of the background $(\gamma_{\mu\nu}\,,\Phi)$, again with detailed radial structure. Similar comments should apply to the perturbative part h_{ijF} of the spatial gravitational field on Σ_F , and to the final spatial spin-1 potential A_{iF} , if appropriate. As a result, in the (complexified) nearly-Lorentzian régime, one will have (classically) radiation of various spins, typically with wavelengths much shorter than the characteristic length- or time-scale corresponding to the Schwarzschild mass M ; this radiation will propagate approximately by geometrical optics. In turn, the effective energy-momentum tensor $T_{\mu\nu}$ due to this radiation will, on the average, be nearly spherically symmetric, and will indeed have the form appropriate to a radially-outgoing null fluid [40,41]. The classical 'space-time' metric resulting from such a null-fluid effective $T_{\mu\nu}$ is precisely of the Vaidya type [41]. This resembles the Schwarzschild geometry, except that the rôle of the Schwarzschild mass M is taken by a mass function $m(t\,,r)$, which varies extremely slowly with respect both to t and to r in the space-time region containing outgoing radiation. The perturbative fields, such as $\phi^{(1)}$, propagate adiabatically in this classical solution.

In Sec.2 we shall describe, in tensor language, the boundary data on the final surface Σ_F which are natural for the $s = 1$ Maxwell and for the $s = 2$ graviton cases. These are, respectively, the Maxwell magnetic field B_i and the magnetic part H_{ik} of the Weyl tensor. In Sec.3 we rephrase the $s = 1$ and $s = 2$ problems in terms of 2-component spinors; the Maxwell field strength is determined by the symmetric spinor $\phi_{AB} = \phi_{(AB)}$ and the Weyl curvature by $\Psi_{ABCD} = \Psi_{(ABCD)}$. In both cases, the natural boundary data involve a 'projection' on symmetric spinors; for $s = 1$ and 2 , the data are related by supersymmetry. In Sec.4, for the Maxwell field, we begin the process of decomposing the classical problem in terms of odd- and even-parity harmonics, following Regge and Wheeler. A similar procedure is carried in Sec.5 for the odd-parity gravitational perturbations. Returning to the Maxwell case in Sec.6, the (magnetic) boundary conditions on Σ_F are described, together with the classical action S^{EM}_{class} as a functional of the final boundary data. Secs.7,8 and 9 are concerned with gravitational perturbations. In Sec.7, the odd-parity $s = 2$ problem is treated roughly by analogy with the Maxwell problem of Sec.6. Even-parity gravitational perturbations are treated in Secs.8 and 9. A preliminary

analysis in Sec.8 leads to the classical action functional $S^{(2)}_{\rm class}$ in Sec.9. The Conclusion is in Sec.10.

2 Boundary data for the Maxwell field and for gravity

The Maxwell contribution to the total Lorentzian action S is

$$S^{EM} \quad = \quad -\,\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,(-g)^{\frac{1}{2}}\,F_{\mu\nu}\,F^{\mu\nu} \quad , \tag{2.1}$$

where $F_{\mu\nu} = F_{[\mu\nu]}$ is the Maxwell field strength, while the space-time metric $g_{\mu\nu}$ is assumed here to have Lorentzian signature, with $g = \det(g_{\mu\nu}) < 0$. The resulting classical Maxwell field equations are

$$\nabla_\mu F^{\mu\nu} \quad = \quad 0 \quad . \tag{2.2}$$

The further condition that $F_{\mu\nu}$ be derivable from a vector potential A_μ, as

$$F_{\mu\nu} \quad = \quad \nabla_\mu A_\nu - \nabla_\nu A_\mu \quad = \quad \partial_\mu A_\nu \,-\, \partial_\nu A_\mu \quad , \tag{2.3}$$

may equivalently be written in the form of the dual field equations

$$\nabla_\mu({}^*F^{\mu\nu}) \quad = \quad 0 \quad , \tag{2.4}$$

where

$${}^*F_{\mu\nu} \quad = \quad \frac{1}{2}\,\epsilon_{\mu\nu}{}^{\rho\sigma}\,F_{\rho\sigma} \tag{2.5}$$

is the dual field strength [42,43]. (The Poincaré lemma [44] may be applied, since we are working within a manifold $\mathcal{M}$ which may be regarded as a slice of $\mathbb{R}^4$, with boundary $\partial\mathcal{M}$ consisting of two $\mathbb{R}^3$ hypersurfaces.) The action (2.1) is invariant under Maxwell gauge transformations

$$A_\mu \quad \to \quad A_\mu \,+\, \partial_\mu\Lambda \quad , \tag{2.6}$$

in the interior, where $\Lambda(x)$ is a function of position.

As in [33,34,45] for scalar (spin-0) perturbations of spherically symmetric Einstein/massless-scalar gravitational collapse, and as in the treatment of spin-2 (graviton) perturbations below, we shall need the classical action $S_{\rm class}$, namely the action S evaluated at a classical solution of the appropriate (slightly complexified) boundary-value problem, as a functional of the boundary data. From this, one obtains the semi-classical quantum amplitude, proportional to $\exp(iS_{\rm class})$, and hence by a limiting procedure the Lorentzian quantum amplitude. In the present (spin-1) Maxwell case, the classical action $S^{EM}_{\rm class}$ resides solely on the boundary $\partial\mathcal{M}$, which consists of the initial space-like hypersurface Σ_I and final hypersurface Σ_F. There will be no contribution from any large cylinder of radius $R_\infty \to \infty$, provided that we impose the physically reasonable

restriction that, as $r \to \infty$, the potential A_μ should die off faster than r^{-1}, and that the field strength $F_{\mu\nu}$ should die off faster than r^{-2}. That is, we impose reasonable fall-off conditions at large r on field configurations, such that the action S should be finite. (Compare the usual fall-off conditions for instantons in Euclidean Yang-Mills theory [44,48,49].) For the above class of Maxwell field configurations, the boundary form of the classical Maxwell action is

$$S^{EM}_{\text{class}} \quad = \quad -\ \frac{1}{8\pi}\int_{\Sigma_I}^{\Sigma_F} d^3x\, h^{\frac{1}{2}}\, n_\mu\, A_\nu\, F^{\mu\nu} \quad . \tag{2.7}$$

Here, as in Sec.1, $h_{ij} = g_{ij}$ $(i,j = 1,2,3)$ gives the intrinsic Riemannian 3-metric on the boundary hypersurface Σ_I or Σ_F, and we write $h = \det(h_{ij}) > 0$. Further, n^μ denotes the (Lorentzian) unit future-directed timelike vector, normal to the space-like hypersurface Σ_I or Σ_F.

Given the 3 + 1 split of the 4-metric $g_{\mu\nu}$ at each boundary, due to the ability to project vectors and tensors normally using n^μ and tangentially using the projector [48]

$$h_{\mu\nu} \quad = \quad g_{\mu\nu} + n_\mu\, n_\nu \quad , \tag{2.8}$$

at the boundary, one can project the potential A_μ and field strength $F_{\mu\nu}$ into 'normal' and 'spatial' parts on Σ_I and Σ_F. In particular, one defines the densitised electric field vector on the boundary:

$$\mathcal{E}^i \quad = \quad -\ h^{\frac{1}{2}} E^i \quad , \tag{2.9}$$

where

$$E^\nu \quad = \quad n_\mu\, F^{\mu\nu} \tag{2.10}$$

obeys $n_\nu E^\nu = 0$. Further, in a Hamiltonian formulation [49], when one regards the spatial components A_i of the vector potential as 'coordinates', then the canonical momentum π^i, automatically a vector density, is given by

$$\pi^i \quad = \quad -\ \frac{\mathcal{E}^i}{4\pi} \quad . \tag{2.11}$$

Note that the normal component $A_t = -\varphi$, where φ is the Maxwell scalar potential, is gauge-dependent, but that φ does not need to be specified on the spacelike boundaries Σ_I and Σ_F, and is indeed allowed to vary freely there and throughout the space-time. Its conjugate momentum therefore vanishes. In the gravitational case, analogous properties hold for the lapse function N and the shift vector N^i [15,49].

As described in [49-51], it is natural in specifying a classical boundary-value problem for the Maxwell field, with data given on the space-like boundaries Σ_I and Σ_F and at spatial infinity (with Lorentzian proper-time separation T), to fix the spatial magnetic field components, described in densitised form by

$$\mathcal{B}^i \quad = \quad \frac{1}{2}\,\epsilon^{ijk}\, F_{jk} \quad , \tag{2.12}$$

on Σ_I and Σ_F . The $\mathcal{B}^i$ cannot be specified freely on the boundary, but are further subject to the (linear) restriction

$$\partial_i \mathcal{B}^i \quad = \quad 0 \quad . \tag{2.13}$$

These components are gauge-invariant, and therefore physically measurable, in contrast to those of the spatially-projected vector potential A_i . We shall regard the space of such $\mathcal{B}^i(x)$, on Σ_I or Σ_F , as the 'coordinates' for Maxwell theory. From the space-time Maxwell equations (2.2), one also deduces the constraint

$$\partial_i \mathcal{E}^i \quad = \quad 0 \quad . \tag{2.14}$$

Turning to $s = 2$ (graviton) perturbations of a spherically-symmetric background, we describe the boundary conditions found below to be appropriate both for odd- and even-parity vacuum $s = 2$ perturbations. The most suitable $s = 2$ boundary data involve prescribing the magnetic part of the Weyl curvature tensor $C_{\alpha\beta\gamma\delta}$ [42,43,52,53] on Σ_I and on Σ_F . For simplicity, as above, we are taking the gravitational initial data on Σ_I to be exactly spherically symmetric ('no incoming gravitons'). Of course, in a large part of the space-time, one is nearly *in vacuo*, the Ricci tensor then obeying $R_{\alpha\beta} \simeq 0$, whence $C_{\alpha\beta\gamma\delta} \simeq R_{\alpha\beta\gamma\delta}$, the Riemann tensor. More generally, the Weyl tensor is defined by

$$R_{\alpha\beta\gamma\delta} \quad = \quad C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha} - \frac{1}{3}\,R\,g_{\alpha[\gamma}g_{\delta]\beta} \quad , \tag{2.15}$$

where $R = g^{\alpha\beta}\,R_{\alpha\beta}$ gives the Ricci scalar, and where square brackets denote anti-symmetrisation.

The algebraic symmetries of the Weyl tensor at a point are summarised by

$$\begin{aligned} C_{\alpha\beta\gamma\delta} \quad &= \quad C_{[\alpha\beta][\gamma\delta]} \quad = \quad C_{\gamma\delta\alpha\beta} \quad , \\ C_{\alpha[\beta\gamma\delta]} \quad &= \quad 0 \quad , \qquad C^{\alpha}{}_{\beta\alpha\delta} \quad = \quad 0 \quad . \end{aligned} \tag{2.16}$$

These imply that $C_{\alpha\beta\gamma\delta}$ has 10 algebraically-independent components at each point. At a bounding space-like hypersurface, such as Σ_F , one can, by analogy with the Maxwell case, apply a 3 + 1 decomposition to the Weyl tensor $C_{\alpha\beta\gamma\delta}$, which splits into two symmetric trace-free spatial tensors, the electric part E_{ik} and the magnetic part H_{ik} of the Weyl tensor [52,53]. Thus, the 10 space-time components of $C_{\alpha\beta\gamma\delta}$ have been decomposed into the 5 spatial components of E_{ik} and 5 more of H_{ik} . (Correspondingly, in Maxwell theory above, the 6 non-trivial components of the field strength $F_{\mu\nu}$ became the 3 of E_i plus the 3 of B_i .)

For convenience of exposition, consider an 'adapted' coordinate system $(x^0\,, x^1\,, x^2\,, x^3)$ in a neighbourhood of Σ_F , such that Σ_F lies at $x^0 = 0$, and such that $n^0 = 1$ at all points of Σ_F . The spatial 3-metric is, as usual, denoted by h_{ij} , and we again write $h = \det(h_{ij})$. The electric part of the Weyl tensor is defined in 4-dimensional language to be

$$E_{\alpha\gamma} \quad = \quad C_{\alpha\beta\gamma\delta}\,n^\beta\,n^\delta \quad . \tag{2.17}$$

In an adapted coordinate system, this corresponds to the 'spatial' equation

$$E_{ik} \quad = \quad C_{i0k0} \quad . \tag{2.18}$$

The magnetic part of the Weyl tensor is defined to be

$$H_{\alpha\gamma} \quad = \quad \frac{1}{2}\,\eta_{\alpha\beta}{}^{\rho\sigma}\,C_{\rho\sigma\gamma\delta}\,n^{\rho}\,n^{\delta} \quad , \tag{2.19}$$

where

$$\eta_{\alpha\beta\gamma\delta} \quad = \quad \eta_{[\alpha\beta\gamma\delta]} \quad = \quad (-g)^{\frac{1}{2}}\,\epsilon_{\alpha\beta\gamma\delta} \tag{2.20}$$

is the alternating tensor, with $g = \det(g_{\mu\nu})$, and where $\epsilon_{\alpha\beta\gamma\delta} = \epsilon_{[\alpha\beta\gamma\delta]}$ is the alternating symbol, normalised such that $\epsilon_{0123} = 1$. In an adapted coordinate system, one finds that

$$H_{ik} \quad = \; -\,\frac{1}{2}\,h^{-1/2}\,h_{in}\,\epsilon^{n\ell m}\,C_{\ell mk0} \quad . \tag{2.21}$$

Both E_{ik} and H_{ik}, so defined, are the components of 3-dimensional (spatial) tensors, obeying the algebraic restrictions (symmetric, traceless):

$$E_{ik} \quad = \quad E_{(ik)} \quad , \qquad h^{ik}\,E_{ik} \quad = \quad 0 \quad ; \tag{2.22}$$

$$H_{ik} \quad = \quad H_{(ik)} \quad , \qquad h^{ik}\,H_{ik} \quad = \quad 0 \quad , \tag{2.23}$$

where round brackets denote symmetrisation. By analogy with the vacuum Maxwell case for E_i, B_i above, here E_{ik} and H_{ik} also obey differential constraints on the bounding 3-surface. From the Bianchi identities [42,43], one has (*in vacuo*)

$$^{3}\nabla_k E^{ik} \quad = \quad 0 \quad , \qquad {}^{3}\nabla_k H^{ik} \quad = \quad 0 \quad , \tag{2.24}$$

where $^{3}\nabla_k$ denotes the intrinsic 3-dimensional covariant derivative, which preserves the 3-metric h_{ij}.

The classical Einstein-Hilbert action functional for gravity, with magnetic data H_{ik} specified on the boundaries, will be discussed in Secs.5,7-9 below, for the case of weak anisotropic perturbations.

3 Boundary conditions in two-component spinor language

As mentioned in the Introduction, a more unified view of the boundary conditions for perturbed data, as specified on the initial and final space-like hypersurfaces Σ_I and Σ_F, can be gained from their description in terms of 2-component spinors [15,42,43]. In this Section, we again begin with $s = 1$ Maxwell perturbations.

Consider (in Lorentzian signature) a real Maxwell field-strength tensor $F_{\mu\nu}$, obeying the Maxwell equations (2.2,4). In the theory of 2-component spinors, a

space-time index μ is related to a pair of spinor indices AA' ($A=0\,,1\,;\ A'=0'\,,1'$) through the (hermitian) spinor-valued 1-forms $e^{AA'}{}_{\mu}$, defined by [15,42,43]:

$$e^{AA'}{}_{\mu} \quad = \quad e^{a}{}_{\mu}\,\sigma_{a}{}^{AA'} \quad . \tag{3.1}$$

Here, $e^{a}{}_{\mu}$ denotes a (pseudo-)orthonormal basis of 1-forms $(a\,=\,0\,,1\,,2\,,3)$, while $\sigma_{a}{}^{AA'}$ denotes the Infeld-van der Waerden translation symbols [15,42,43]. At this point, one has to make a definite decision between $SL(2,\mathbb{C})$ spinors, which are the most appropriate for studying real Lorentzian-signature geometry, and $SO(4)$ spinors, in which Riemannian geometry is most simply described [54]. We follow the Lorentzian spinor conventions of [15]. This does not prevent one from describing complex or Riemannian geometry – one simply allows the 'tetrad' $e^{a}{}_{\mu}$ of 1-forms above to become suitably complex.

Knowledge of $F_{\mu\nu}$ at a point is equivalent to knowledge of

$$F_{AA'BB'} \quad = \quad F_{\mu\nu}\,e_{AA'}{}^{\mu}\,e_{BB'}{}^{\nu} \tag{3.2}$$

at that point. Here, $F_{AA'BB'}$ is hermitian, for real $F_{\mu\nu}$. Further, the space-time antisymmetry $F_{\mu\nu}\,=\,F_{[\mu\nu]}$ implies that the decomposition

$$F_{AA'BB'} \quad = \quad \epsilon_{AB}\,\tilde{\phi}_{A'B'}\,+\,\epsilon_{A'B'}\,\phi_{AB} \tag{3.3}$$

holds, where ϵ_{AB} and $\epsilon_{A'B'}$ are the unprimed and primed alternating spinors [42,43], and where

$$\phi_{AB} \quad = \quad \frac{1}{2}\,F_{AA'B}{}^{A'} \quad = \quad \phi_{BA} \tag{3.4}$$

is a symmetric spinor. In the present Lorentzian case with real Maxwell field, $\tilde{\phi}_{A'B'}$ is the spinor hermitian-conjugate to ϕ_{AB}. (In the Riemannian context, Eq.(3.3) would give the splitting of the Maxwell field strength into its self-dual and anti-self-dual parts [26,44].) Knowledge of the 3 complex components of ϕ_{AB} at a point is equivalent to knowledge of the 6 real components of $F_{\mu\nu}$ at that point; also, the ϕ_{AB} are, in principle, physically measurable, just as the $F_{\mu\nu}$ are.

In terms of the dual field strength ${}^{*}F_{\mu\nu}$ of Eq.(2.5), one finds that

$$F_{\mu\nu}\,+\,i\,{}^{*}F_{\mu\nu} \quad = \quad 2\,\phi_{AB}\,\epsilon_{A'B'}\,e^{AA'}{}_{\mu}\,e^{BB'}{}_{\nu} \quad , \tag{3.5}$$

together with the conjugate equation. The vacuum Maxwell field equations (2.2,4) can then be combined to give

$$\nabla^{AA'}\phi_{AB} \quad = \quad 0 \quad , \qquad \nabla^{AA'}\tilde{\phi}_{A'B'} \quad = \quad 0 \quad , \tag{3.6}$$

where $\nabla^{AA'}\,=\,e^{AA'\mu}\,\nabla_{\mu}$.

Here, we are again interested in the decomposition of the Maxwell field strength with respect to a space-like bounding hypersurface and its associated unit (future-directed) normal vector n^{μ}. Define the normal spinor

$$n^{AA'} \quad = \quad n^{\mu}\,e^{AA'}{}_{\mu} \quad . \tag{3.7}$$

Then the (purely spatial) electric and magnetic field vectors E_k and B_k can be expressed through

$$E_k + i\,B_k \quad = \quad 2\,\phi_{AB}\,n^A{}_{B'}\,e^{BB'}{}_k \quad , \tag{3.8}$$
$$E_k - i\,B_k \quad = \quad 2\,\bar{\phi}_{A'B'}\,n_B{}^{A'}\,e^{BB'}{}_k \quad . \tag{3.9}$$

In 4-vector language, the corresponding co-vector fields E_μ and B_μ are defined by

$$E_\mu \quad = \quad n^\nu\,F_{\nu\mu} \quad , \qquad B_\mu \quad = \quad n^\nu\,{}^*F_{\nu\mu} \quad , \tag{3.10}$$

obeying

$$n^\mu\,E_\mu \quad = \quad 0 \quad = \quad n^\mu\,B_\mu \quad . \tag{3.11}$$

Next, for $\epsilon \;=\; \pm 1$, define

$$\Psi_\epsilon^{AB} \quad = \quad 2\,\epsilon\,n^A{}_{A'}\,n^B{}_{B'}\,\tilde{\phi}^{A'B'} + \phi^{AB} \quad . \tag{3.12}$$

Here, Ψ_ϵ^{AB} is symmetric on A and B ; this spinor may be re-expressed in terms of E_k and B_k , on making use of the symmetry of $n_B{}^{B'}\,e^{BA'k}$ on its free spinor indices $B'A'$ [15]. Here, we define

$$e^{BA'k} \quad = \quad h^{k\ell}\,e^{BA'}{}_\ell \quad , \tag{3.13}$$

where $h^{k\ell}$ is the inverse spatial metric. The above symmetry property then reads

$$n_B{}^{B'}\,e^{BA'k} \quad = \quad n_B{}^{A'}\,e^{BB'k} \quad . \tag{3.14}$$

From Eq.(3.14), we find the decomposition

$$\Psi_\epsilon^{AB} \quad = \quad n^B{}_{B'}\,e^{AB'k}\left[(\epsilon - 1)E_k - i\,(\epsilon + 1)\,B_k\right] \quad . \tag{3.15}$$

In particular, our boundary condition in Sec.6 below of fixing the magnetic field (a spatial co-vector field) on each of the initial and final space-like hypersurfaces Σ_I and Σ_F is equivalent to fixing the spinorial expression

$$\Psi_+^{AB} \quad = \quad -\,2\,i\,n^B{}_{B'}\,e^{AB'k}\,B_k \tag{3.16}$$

on each boundary. Note that, even though we regard B_k as having 3 real components, the left-hand side, being symmetric on (AB) , appears to have 3 complex components. In fact, Ψ_+^{AB} as defined through Eq.(3.16) obeys a further hermiticity requirement, appropriate for spinors in 3 Riemannian dimensions (that is, on the hypersurfaces Σ_I and Σ_F) [15,42,43], so re-balancing matters.

For comparison with much of the work done on black holes and their perturbations, one needs the Newman-Penrose formalism [55] – an essentially spinorial description of the geometry. Here, considering only unprimed spinors at present, a normalised dyad $(o^A\,,\,\iota^A)$ at a point is defined to be a basis for the 2-complex-dimensional vector space of spinors ω^A at that point, normalised according to

$$o_A\,\iota^A \quad = \quad 1 \quad = \quad -\,\iota_A\,o^A \quad . \tag{3.17}$$

The unprimed field strength $\phi_{AB} = \phi_{(AB)}$ can be projected onto the dyad, to give the 3 Newman-Penrose quantities

$$\phi_0 \;=\; \phi_{AB}\, o^A\, o^B \quad , \quad \phi_1 \;=\; \frac{1}{2}\,\phi_{AB}\, o^A\, \iota^B \quad , \quad \phi_2 \;=\; \phi_{AB}\, \iota^A\, \iota^B \quad , \tag{3.18}$$

each of which is a complex scalar field (function). Using the Newman-Penrose formalism to describe perturbations in the background of a rotating Kerr black-hole geometry, Teukolsky [56] derived decoupled separable equations for the quantities ϕ_0 $(s = 1)$ and $r^2\,\phi_2$ $(s = -1)$ (for further review, see [39,57].) In our non-rotating case, with spherically-symmetric background, the Newman-Penrose quantity of most interest to us, following the work of this paper, is ϕ_1. In the language of [55,58], ϕ_1 has spin and conformal weight zero. This is best described in the Kinnersley null tetrad [59] for the Schwarzschild or Kerr geometry, in our coordinate system. A null tetrad [55] ℓ^μ, n^μ, m^μ, $\bar{m}^\mu$ of vectors at a point is a set obeying, in an obvious notation, $\ell.\ell = n.n = m.m = \bar{m}.\bar{m} = 0$; $\ell.n = -\,m.\bar{m} = 1$; $\ell.m = \ell.\bar{m} = n.m = n.\bar{m} = 0$. Knowledge of such a null tetrad is equivalent to knowledge of the corresponding normalised spinor dyad $(o^A\,,\,\iota^A)$, through the relations

$$\begin{array}{ccccccc} l^\mu & \leftrightarrow & o^A\, o^{A'} & , & n^\mu & \leftrightarrow & \iota^A\, \iota^{A'} \quad , \\ m^\mu & \leftrightarrow & o^A\, \iota^{A'} & , & \bar{m}^\mu & \leftrightarrow & \iota^A\, o^{A'} \quad . \end{array} \tag{3.19}$$

In terms of the Regge-Wheeler variables to be used for the decomposition of the linearised Maxwell field strength $F^{(1)}_{\mu\nu}$ given in Secs.4,6, one has

$$\phi_1 \quad = \quad \frac{1}{2r^2}\sum_{\ell m}\Big(\psi^{(e)}_{1\ell m} + i\,\psi^{(o)}_{1\ell m}\Big)\, Y_{\ell m}(\Omega) \quad , \tag{3.20}$$

where the $Y_{\ell m}(\Omega)$ are the normalised spherical harmonics of [60]. As a result, one finds that $r^2\,\phi_1$ obeys the wave equation (4.47) below. With regard to the boundary conditions on Σ_I and Σ_F, when the variables $\psi^{(e)}_{1\ell m}$ and $\psi^{(o)}_{1\ell m}$ are being used, the correct boundary data (Sec.6) will involve specifying both $\psi^{(o)}_{1\ell m}$ and $\partial_t\psi^{(e)}_{1\ell m}$ on Σ_I and Σ_F.

Turning again to the gravitational field, in 2-component spinor language [42,43], one has the decomposition of the Weyl tensor:

$$C_{\alpha\beta\gamma\delta} \quad \leftrightarrow \quad \epsilon_{AB}\,\epsilon_{CD}\,\tilde{\Psi}_{A'B'C'D'} + \epsilon_{A'B'}\,\epsilon_{C'D'}\,\Psi_{ABCD} \quad , \tag{3.21}$$

where6

$$\Psi_{ABCD} \quad = \quad \Psi_{(ABCD)} \tag{3.22}$$

is the totally symmetric (complex) Weyl spinor, and (in Lorentzian signature) $\tilde{\Psi}_{A'B'C'D'}$ is its hermitian conjugate. The dual of the Weyl tensor is defined as

$${}^*C_{\alpha\beta\gamma\delta} \;=\; \frac{1}{2}\,\eta_{\alpha\beta\rho\sigma}\, C^{\rho\sigma}{}_{\gamma\delta} \quad . \tag{3.23}$$

One finds that:

$$\Bigl(C_{\alpha\beta\gamma\delta} + i\,{}^{*}C_{\alpha\beta\gamma\delta}\Bigr) \quad \leftrightarrow \quad 2\,\epsilon_{A'B'}\,\epsilon_{C'D'}\,\Psi_{ABCD} \quad , \tag{3.24}$$

together (in Lorentzian signature) with the hermitian-conjugate equation. (If instead we had used the Euclidean definition of spinors, then Eq.(3.21) would describe the splitting of the Weyl tensor into self-dual and anti-self-dual parts [44,58].) The (vacuum) Bianchi identities read [42,43]:

$$\nabla^{AA'}\,\Psi_{ABCD} \quad = \quad 0 \quad , \qquad \nabla^{AA'}\,\tilde{\Psi}_{A'B'C'D'} \quad = \quad 0 \quad . \tag{3.25}$$

On the bounding surface Σ_F (say), one finds analogously that

$$E_{k\ell} + i\,H_{k\ell} \quad = \quad 2\,\Psi_{ABCD}\,\bigl(n^{A}{}_{B'}\,e^{BB'}{}_{k}\bigr)\bigl(n^{C}{}_{D'}\,e^{DD'}{}_{\ell}\bigr) \tag{3.26}$$

and its hermitian conjugate. Thence, the magnetic tensor $H_{k\ell}$ is given by

$$H_{k\ell} \quad = \quad \Bigl(-\,i\,\Psi_{ABCD}\Bigl(n^{A}{}_{B'}\,e^{BB'}{}_{k}\Bigr)\Bigl(n^{C}{}_{D'}\,e^{DD'}{}_{\ell}\Bigr)\Bigr) + \Bigl[\text{h.c.}\Bigr] \quad , \tag{3.27}$$

with a corresponding equation for $E_{k\ell}$. These two equations can straightforwardly be inverted to give an expression analogous to Ψ_{+}^{AB} for the $s=1$ Maxwell case of Eq.(3.16). This analogous expression, $\Psi_{+}^{ABCD} = \Psi_{+}^{(ABCD)}$, is again totally symmetric on its indices, and is given for $\epsilon = \pm 1$ by the generalisation of Eq.(3.12), as

$$\Psi_{\epsilon}^{ABCD} \quad = \quad 4\,\epsilon\,n^{A}{}_{A'}\,n^{B}{}_{B'}\,n^{C}{}_{C'}\,n^{D}{}_{D'}\,\tilde{\Psi}^{A'B'C'D'} + \Psi^{ABCD} \quad . \tag{3.28}$$

Thus, again for the magnetic case $\epsilon = +1$, Ψ_{+}^{ABCD} provides a spinorial version of the magnetic part $H_{k\ell}$ of the Weyl tensor, to be fixed on the final boundary Σ_F in our $s=2$ treatment of the gravitational boundary-value problem, perturbed about spherically-symmetric collapse. Of course, the perturbative boundary data $H_{k\ell}$ must further be chosen such that the divergence condition (2.24) holds: ${}^{3}\nabla_k H^{ik} = 0$ on Σ_F, just as, from Eq.(2.13) in the Maxwell case, the condition ${}^{3}\nabla_k B^k = 0$ must hold.

Two of the five complex components of Ψ_{ABCD} are contained in the Newman-Penrose quantities [42,43,55]

$$\Psi_0 \quad = \quad \Psi_{ABCD}\,o^A\,o^B\,o^C\,o^D \quad , \qquad \Psi_4 \quad = \quad \Psi_{ABCD}\,\iota^A\,\iota^B\,\iota^C\,\iota^D \quad , \tag{3.29}$$

where $(o^A\,,\,\iota^A)$ is a normalised spinor dyad [43,43,55]. Again taking the Kinnersley null tetrad [39,59], it was further shown by Teukolsky [56] that Ψ_0 and Ψ_4 each obey decoupled separable wave equations. Following work by Chrzanowski [61], it was confirmed by Wald [62] that, given a solution Ψ_0 or Ψ_4 of the Teukolsky equation for a (nearly-) Kerr background, all the vacuum metric perturbations can be reconstructed in a certain gauge through a series of simple

linear operations on Ψ_0 (or on Ψ_4) [39,61,62]. Once the linearised metric perturbations are known, then, of course, one can compute other Newman-Penrose quantities at linearised order, such as

$$\Psi_2 \quad = \quad \Psi_{ABCD}\, o^A\, o^B\, \iota^C\, \iota^D \quad . \tag{3.30}$$

Note here that, for an unperturbed Schwarzschild background (say), only the middle Newman-Penrose quantity Ψ_2, out of the set Ψ_i $(i = 0, \ldots, 4)$, is non-zero, with $\Psi_2 = -M/r^3$ in Schwarzschild coordinates [57].

The analogous process is implicit in the (Maxwell) discussion above, leading to Eq.(3.20): For $s = 1$ perturbations of the Kerr metric, the corresponding linearised Maxwell vector potential A_μ (in a particular gauge) can be reconstructed by simple steps from the Newman-Penrose quantites ϕ_0 or ϕ_2 [39,61,62]. Hence, the middle Newman-Penrose quantity ϕ_1 can also be found, leading to Eq.(3.20).

In the more complicated $s = 2$ case, although we have not yet finished detailed calculations on this point, it does now look reasonable to expect that, for gravitational perturbations about a spherically-symmetric gravitational collapse, there should exist a relation analogous to the $s = 1$ Eq.(3.20). In this case, for a Schwarzschild background, this relation would involve expanding out the first-order perturbations of the middle Newman-Penrose quantity Ψ_2 in terms of the $s = 2$ Regge-Wheeler description of Secs.5,7-9 below.

4 Regge-Wheeler formalism – Maxwell case

A more unified treatment of the angular harmonics appearing in the separation process for $s = 1$ (Maxwell) and $s = 2$ (graviton) perturbations of a spherically-symmetric background can be given in terms of vector and tensor harmonics [63]. In [11], we expanded the $s = 0$ (massless-scalar) perturbations in terms of scalar spherical harmonics $Y_{\ell m}(\theta, \phi)$, which have even parity. Vector and tensor spherical harmonics, however, can have odd as well as even parity.

Any vector field in a spherically-symmetric background, such as the classical $s = 1$ (photon) solutions appearing in the background gravitational-collapse geometry , can be expanded in terms of vector spherical harmonics on the unit 2-sphere [37,38]. Correspondingly, angular vector and tensor indices are raised and lowered with the metric $\hat{\gamma}_{ab}$, given by

$$\hat{\gamma}_{\theta\theta} \;=\; 1 \quad , \qquad \hat{\gamma}_{\phi\phi} \;=\; \sin^2\theta \quad , \qquad \hat{\gamma}_{\theta\phi} \;=\; \hat{\gamma}_{\phi\theta} \;=\; 0 \quad . \tag{4.1}$$

The even-parity vector harmonics [37,38] have angular components

$$(\Psi_{\ell m})_a \quad = \quad \big(\partial_a Y_{\ell m}\big) \quad , \tag{4.2}$$

where $a = (\theta, \phi)$. The odd-parity vector harmonics are

$$(\Phi_{\ell m})_a \quad = \quad \epsilon_a{}^b\, \big(\partial_b Y_{\ell m}\big) \quad . \tag{4.3}$$

Here, ${\epsilon_a}^{b}$ denotes the tensor with respect to angular indices $(a = \theta\,,\phi\,;\ b = \theta\,,\phi)$, such that the lowered version $\epsilon_{ab} = -\,\epsilon_{ba}$ is anti-symmetric, with $\epsilon_{01} = \big(\hat{\gamma}\big)^{\frac{1}{2}} = \sin\theta$, where $\hat{\gamma} = \det\big(\hat{\gamma}_{ab}\big)$. Thus,

$$
{\epsilon_\theta}^{\phi} \;=\; \frac{-1}{\sin\theta} \quad , \qquad {\epsilon_\phi}^{\theta} \;=\; \sin\theta \quad , \qquad {\epsilon_\phi}^{\phi} \;=\; {\epsilon_\theta}^{\theta} \;=\; 0 \quad . \tag{4.4}
$$

The forms of the angular harmonics appearing in the $s = 1$ photon calculations below can be deduced from these vector-spherical-harmonic expressions.

Analogously, any rank-2 tensor field such as a linearised (graviton) $s = 2$ classical solution, to be treated in Secs.5,7-9, below, can be expanded in terms of tensor spherical harmonics. The even-parity tensor harmonics are

$$
(\Psi_{\ell m})_{ab} \;=\; Y_{\ell m|ab} \quad , \qquad (\Phi_{\ell m})_{ab} \;=\; \hat{\gamma}_{ab}\,Y_{\ell m} \quad , \tag{4.5}
$$

where a bar $|$ denotes a covariant derivative with respect to the metric $\hat{\gamma}_{ab}$. The odd-parity tensor harmonics are

$$
(\chi_{\ell m})_{ab} \;=\; \frac{1}{2}\Big[{\epsilon_a}^{c}\Big(\Psi_{\ell m}\Big)_{cb} + {\epsilon_b}^{c}\Big(\Psi_{\ell m}\Big)_{ca}\Big] \quad . \tag{4.6}
$$

From 1957, Regge and Wheeler [63] developed the formalism for treating both spin-1 and spin-2 classical perturbations of the Schwarzschild solution and of other spherically-symmetric solutions, corresponding to Maxwell and gravitational (graviton) perturbations. Here, in the Regge-Wheeler (RW) formalism, for Maxwell theory we decompose the real linearised field strength $F^{(1)}_{\mu\nu}$ and linearised vector potential $A^{(1)}_{\mu}$ into tensor and vector spherical harmonics, respectively [37,38]. We are assuming that the background (unperturbed) classical solution consists, as above, of a spherically-symmetric gravitational and massless-scalar field ($\gamma_{\mu\nu}\,,\ \Phi$), with no background Maxwell field: $A^{(0)}_{\mu} = 0\,,\ F^{(0)}_{\mu\nu} = 0\,.$

For each spin $s = 0\,,1\,,2\,,$ the corresponding perturbation modes split into those with even parity and those with odd parity. Under the parity inversion: $\theta \to (\pi - \theta)\,,\ \phi \to (\pi + \phi)\,,$ we define the even perturbations as those with parity $\pi = (-1)^{\ell}$, while the odd perturbations have parity $\pi = (-1)^{\ell+1}$. For Maxwell theory $(s = 1)$, the $\ell = 0$ mode corresponds to a static perturbation, in which a small amount of electric charge is added to the black hole; in particular, a Schwarzschild solution will be 'displaced' infinitesimally along the family of Reissner-Nordström solutions. For radiative modes with $\ell = 1$ (dipole) and higher, we set

$$
F^{(1)}_{\mu\nu}(x) \;=\; \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\Big[\Big(F^{(o)}_{\mu\nu}\Big)_{\ell m} + \Big(F^{(e)}_{\mu\nu}\Big)_{\ell m}\Big] \quad , \tag{4.7}
$$

$$
A^{(1)}_{\mu}(x) \;=\; \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\Big[\Big(A^{(o)}_{\mu}\Big)_{\ell m} + \Big(A^{(e)}_{\mu}\Big)_{\ell m}\Big] \quad . \tag{4.8}
$$

On substituting this decomposition into the boundary expression (2.7) for the classical Maxwell action $S^{EM}_{\rm class}$, we find

$$S^{EM}_{\rm class} = -\,\frac{1}{8\pi}\sum_{\ell m\ell' m'}\int d\Omega\int_0^{R_\infty} dr\, r^2\, e^{(a-b)/2}\,\gamma^{ij}\Big(A^{(o)}_j\Big)_{\ell m}\Big(F^{(o)}_{ti}\Big)^*_{\ell' m'}\Bigg|^{\Sigma_F}_{\Sigma_I}$$
$$-\,\frac{1}{8\pi}\sum_{\ell m\ell' m'}\int d\Omega\int_0^{R\infty} dr\, r^2\, e^{(a-b)/2}\,\gamma^{ij}\Big(A^{(e)}_j\Big)_{\ell m}\Big(F^{(e)}_{ti}\Big)^*_{\ell' m'}\Bigg|^{\Sigma_F}_{\Sigma_I}\,, \tag{4.9}$$

where Ω denotes the angular coordinates $\theta\,,\phi$, and we work at present with the Lorentzian form of the spherically-symmetric background metric (for which the Riemannian form was given in Eq.(1.7), by suitable choice of coordinates), writing:

$$ds^2 \;=\; -\;e^{b(t,r)}\,dt^2 +\; e^{a(t,r)}\,dr^2 +\; r^2\,(d\theta^2 +\;\sin^2\theta\;d\phi^2)\;. \tag{4.10}$$

For later reference, we introduce a standard expression for such a spherically-symmetric space-time, defining the 'mass function' $m(t\,,r)$ by

$$e^{-a} \;=\; 1\,-\,\frac{2m(t\,,r)}{r}\quad , \tag{4.11}$$

within the Vaidya-like region of the space-time, in which the black hole is evaporating. Clearly, the odd and even contributions decouple in the action of Eq.(4.9).

For the subsequent detailed treatment of the angular harmonics involved, we follow Zerilli's decomposition [64] of $F^{(1)}_{\mu\nu}$ and $A^{(1)}_\mu$. Thus, we set

$$F^{(1)}_{\mu\nu}(x) \;=\; \sum_{\ell m}\Big(F^{(1)}_{\mu\nu}\Big)_{\ell m}\quad , \tag{4.12}$$

$$\Big(F^{(1)}_{\mu\nu}\Big)_{\ell m} \;=\; \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ -f_1 & 0 & f_4 & f_5 \\ -f_2 & -f_4 & 0 & f_6 \\ -f_3 & -f_5 & -f_6 & 0 \end{pmatrix}\quad . \tag{4.13}$$

For a given choice of $(\ell\,,m)$, we take

$$f_1 \;=\; \Big(\hat F^{(e)}_{tr}\Big)_{\ell m} Y_{\ell m}(\Omega)\quad , \tag{4.14}$$
$$f_2 \;=\; \big(\sin\theta\big)^{-1}\Big(\hat F^{(o)}_{t\theta}\Big)_{\ell m}\big(\partial_\phi Y_{\ell m}\big) + \Big(\hat F^{(e)}_{t\theta}\Big)_{\ell m}\big(\partial_\theta Y_{\ell m}\big)\quad , \tag{4.15}$$
$$f_3 \;=\; -\,\Big(\hat F^{(o)}_{t\theta}\Big)_{\ell m}\big(\sin\theta\big)\big(\partial_\theta Y_{\ell m}\big) + \;\Big(\hat F^{(e)}_{t\theta}\Big)_{\ell m}\big(\partial_\phi Y_{\ell m}\big)\quad , \tag{4.16}$$
$$f_4 \;=\; \big(\sin\theta\big)^{-1}\Big(\hat F^{(o)}_{r\theta}\Big)_{\ell m}\big(\partial_\phi Y_{\ell m}\big) + \Big(\hat F^{(e)}_{r\theta}\Big)_{\ell m}\big(\partial_\theta Y_{\ell m}\big)\quad , \tag{4.17}$$
$$f_5 \;=\; -\,\Big(\hat F^{(o)}_{r\theta}\Big)_{\ell m}\big(\sin\theta\big)\big(\partial_\theta Y_{\ell m}\big) + \Big(\hat F^{(e)}_{r\theta}\Big)_{\ell m}\big(\partial_\phi Y_{\ell m}\big)\quad , \tag{4.18}$$
$$f_6 \;=\; \Big(\hat F^{(o)}_{\theta\phi}\Big)_{\ell m}\big(\sin\theta\big)\;Y_{\ell m}(\Omega)\quad . \tag{4.19}$$

Here, the $Y_{\ell m}(\Omega)$ are scalar spherical harmonics [60], and a caret indicates that the quantity is a function of t and r only.

Again, following [63], for the vector potential we set

$$\left(A_\mu^{(o)}\right)_{\ell m}(x) = \left(0, 0, \frac{a_{2\ell m}(t,r)\big(\partial_\phi Y_{\ell m}\big)}{\ell(\ell+1)\sin\theta}, \frac{-\,a_{2\ell m}(t,r)\big(\sin\theta\big)\big(\partial_\theta Y_{\ell m}\big)}{\ell(\ell+1)}\right), \tag{4.20}$$

$$\left(A_\mu^{(e)}\right)_{\ell m}(x) = \Big(-\,a_{0\ell m}(t,r)\,Y_{\ell m}(\Omega)\,,\; a_{1\ell m}(t,r)\;Y_{\ell m}(\Omega)\,,\;0\;,\;0\Big)\;. \tag{4.21}$$

Now Eq.(4.9) can be expanded out in the form:

$$\begin{aligned} S^{EM}_{\text{class}} \;=\; & -\,\frac{1}{8\pi}\sum_{\ell m\ell' m'}\int d\Omega\int_0^{R\infty} dr\; e^{(a-b)/2} \quad\times \\ & \times\;\left[\left(A_\theta^{(o)}\right)_{\ell m}\left(F_{t\theta}^{(o)}\right)^*_{\ell' m'} + \frac{\left(A_\phi^{(o)}\right)_{\ell m}\left(F_{t\phi}^{(o)}\right)^*_{\ell' m'}}{\sin^2\theta}\right]\Bigg|_{\Sigma_I}^{\Sigma_F} \\ & -\,\frac{1}{8\pi}\sum_{\ell m\ell' m'}\int d\Omega\int_0^{R\infty} dr\; r^2\,e^{(a-b)/2}\left(A_r^{(e)}\right)_{\ell m}\left(F_{t,r}^{(e)}\right)^*_{\ell' m'}\Bigg|_{\Sigma_I}^{\Sigma_F}\;. \end{aligned} \tag{4.22}$$

Of course, the components of the field strength are given in terms of those of the vector potential by Eq.(2.3); for example, $F_{t\theta}^{(1)} = \partial_t A_\theta^{(1)} - \partial_\theta A_t^{(1)}$. This gives the relations

$$\left(\hat F_{t\theta}^{(o)}\right)_{\ell m} = \frac{\big(\partial_t a_{2\ell m}\big)}{\ell(\ell+1)}\quad, \tag{4.23}$$

$$\left(\hat F_{t\theta}^{(e)}\right)_{\ell m} = a_{0\ell m}\quad, \tag{4.24}$$

$$\left(\hat F_{r\theta}^{(o)}\right)_{\ell m} = \frac{\big(\partial_r a_{2\ell m}\big)}{\ell(\ell+1)}\quad, \tag{4.25}$$

$$\left(\hat F_{r\theta}^{(e)}\right)_{\ell m} = a_{1\ell m}\quad, \tag{4.26}$$

$$\left(\hat F_{tr}^{(e)}\right)_{\ell m} = \big(\partial_r a_{0\ell m}\big) - \big(\partial_t a_{1\ell m}\big)\quad, \tag{4.27}$$

$$\left(\hat F_{\theta\phi}^{(o)}\right)_{\ell m} = a_{2\ell m}\quad. \tag{4.28}$$

The action (4.22) then simplifies to give

$$\begin{aligned} S^{EM}_{\text{class}} \;=\; & -\,\frac{1}{8\pi}\sum_{\ell m}\ell(\ell+1)\int_0^{R\infty} dr\,e^{(a-b)/2}\,f_{\ell m}^{(o)}\Big(\partial_t f_{\ell m}^{(o)*}\Big)\Bigg|_{\Sigma_I}^{\Sigma_F} \\ & -\,\frac{1}{8\pi}\sum_{\ell m}\int_0^{R\infty} dr\,r^2\,e^{-(a+b)/2}\,a_{1\ell m}\Big(\big(\partial_t a_{1\ell m}^*\big) - \big(\partial_r a_{0\ell m}^*\big)\Big)\Bigg|_{\Sigma_I}^{\Sigma_F}\;, \end{aligned} \tag{4.29}$$

where

$$f_{\ell m}^{(o)} = \frac{a_{2\ell m}}{\ell(\ell+1)} \tag{4.30}$$

determines the odd-parity Maxwell tensor *via* Eqs.(4.23,25,28).

The form of the classical action (4.29) can be further simplified by using the Maxwell field equations (2.2,4). This will lead finally to the form (4.45) below, in which $S^{EM}_{\rm class}$ is expressed explicitly in terms of boundary data, as needed in the subsequent calculation of the quantum amplitude (see [11,16] for the spin-0 analogue).

The linearised Maxwell equations can be written as [49]:

$$F^{(1)\mu\nu}{}_{;\nu} \quad = \quad (-\gamma)^{-\frac{1}{2}}\,\partial_\nu\Big[(-\gamma)^{\frac{1}{2}}\,F^{(1)\mu\nu}\Big] \quad = \quad 0 \quad . \tag{4.31}$$

The $\mu\,=\,t\,,r$ equations give

$$\partial_r\bigg[r^2\Big(\hat{F}^{(e)}_{tr}\Big)_{\ell m}\bigg] - \,\ell(\ell+1)\,e^a\Big(\hat{F}^{(e)}_{t\theta}\Big)_{\ell m} \quad = \quad 0 \quad , \tag{4.32}$$

$$e^a\,\partial_t\Big(\hat{F}^{(e)}_{t\theta}\Big)_{\ell m} - \partial_r\bigg[e^{-a}\Big(\hat{F}^{(e)}_{r\theta}\Big)_{\ell m}\bigg] \quad = \quad 0 \quad , \tag{4.33}$$

$$\partial_r\bigg[e^{-a}\Big(\hat{F}^{(o)}_{r\theta}\Big)_{\ell m}\bigg] - e^a\,\partial_t\Big(\hat{F}^{(o)}_{t\theta}\Big)_{\ell m} - r^{-2}\Big(\hat{F}^{(o)}_{\theta\phi}\Big)_{\ell m} \quad = \quad 0 \quad , \tag{4.34}$$

$$\partial_t\Big(\hat{F}^{(e)}_{tr}\Big)_{\ell m} - \frac{e^{-a}\,\ell(\ell+1)}{r^2}\Big(\hat{F}^{(e)}_{r\theta}\Big)_{\ell m} \quad = \quad 0 \quad . \tag{4.35}$$

The $\mu=\theta\,,\,\phi$ components give the same equations. Note that Eq.(4.32) is just the (source-free) constraint equation $\partial_i\mathcal{E}^{(1)i}\;=\;0$ of Eq.(2.14).

The equations (4.23,25,28) together imply the decoupled wave equation for odd perturbations:

$$(\partial_{r^*})^2\,a_{2\ell m} - (\partial_t)^2\,a_{2\ell m} - V_{1\ell}(r)\,a_{2\ell m} \quad = \quad 0 \quad , \tag{4.36}$$

where

$$V_{1\ell}(r) \quad = \quad \frac{e^{-a}\,\ell(\ell+1)}{r^2} \quad > \quad 0 \tag{4.37}$$

is the (massless) spin-1 effective potential and where, as usual [63], we write $\partial_{r^*}\;=\;e^{-a}\,\partial_r$. If the geometry were exactly Schwarzschild, then the coordinate r^* so defined would be the Regge-Wheeler or 'tortoise' radial coordinate, given by [49,63]:

$$r^* \;\;=\;\; r\,+\,2M\ln(r\,-\,2M) \quad . \tag{4.38}$$

As in [11,16], we assume that the adiabatic approximation is valid in a neighbourhood of the initial and final surfaces, Σ_I and Σ_F . In that case, we can, as before, effectively work with the field equations on a Schwarzschild background, except that the Schwarzschild mass M_0 is replaced by the mass function $m(t\,,r)$, as defined in Eq.(4.11), where $m(t\,,r)$ varies extremely slowly with respect both to time and to radius.

Equation (4.33) gives $\partial_t(\hat{F}^{(e)}_{t\theta})_{\ell m}$ also in terms of $(\hat{F}^{(e)}_{r\theta})_{\ell m}$, while Eq.(4.35) gives $\partial_t(\hat{F}^{(e)}_{tr})_{\ell m}$ also in terms of $(\hat{F}^{(e)}_{r\theta})_{\ell m}$. Together, Eqs.(4.27,33,35) imply that

$$(\partial_{r^*})^2 f^{(e)}_{\ell m} \,-\, (\partial_t)^2 f^{(e)}_{\ell m} \,-\, V_{1\ell}\; f^{(e)}_{\ell m} \quad = \quad 0 \quad , \tag{4.39}$$

where we define

$$f^{(e)}_{\ell m} \;=\; e^{-a}\, a_{1\ell m} \quad . \tag{4.40}$$

Thus, with a suitably defined variable $f^{(e)}_{\ell m}$, the even perturbations obey the same decoupled wave equation (4.36) as the odd perturbations.

Finally [65], we set

$$\psi^{(e)}_{\ell m}(t\,,r) \;=\; \frac{r^2}{\ell(\ell+1)}\Big(\big(\partial_t a_{1\ell m}\big) - \big(\partial_r a_{0\ell m}\big)\Big) \quad , \tag{4.41}$$

which is clearly gauge-invariant. Now $\psi^{(e)}_{\ell m}$ is related simply to the (even-parity) function $f^{(e)}_{\ell m}$ of the previous paragraph: Eqs.(4.33,39) imply that

$$\big(\partial_t \psi^{(e)}_{\ell m}\big) \;=\; -\, f^{(e)}_{\ell m} \quad . \tag{4.42}$$

Equations (4.40,41) can now be used to simplify the classical Lorentzian action (4.29). For ease of comparison with the (second-variation) classical spin-2 action, where the pattern is similar, we define

$$\psi^{(o)}_{1\ell m} \;=\; a_{2\ell m}(t\,,r) \quad , \tag{4.43}$$

$$\psi^{(e)}_{1\ell m} \;=\; \ell(\ell+1)\,\psi^{(e)}_{\ell m}(t\,,r) \quad . \tag{4.44}$$

Then, given weak-field Maxwell boundary data, specified by the linearised magnetic-field mode components $\{B^{(1)i}_{\ell m}\}$ on each of the boundaries Σ_I and Σ_F, the corresponding classical Maxwell action is

$$S^{EM}_{\rm class}\Big[\{B^{(1)i}_{\ell m}\}\Big] = \frac{1}{8\pi}\sum_{\ell m}\frac{(\ell-1)!}{(\ell+1)!}\int_0^{R\infty} dr\, e^a\Big(\psi^{(e)}_{1\ell m}\big(\partial_t\psi^{(e)*}_{1\ell m}\big) - \psi^{(o)}_{1\ell m}\big(\partial_t\psi^{(o)*}_{1\ell m}\big)\Big)\Bigg|^{\Sigma_F}_{\Sigma_I} . \tag{4.45}$$

Of course, the limit $R_\infty \to \infty$ must be understood in Eq.(4.45).

Note further that, from Eqs.(4.36,39), one has

$$\big(\partial_{r*}\,\psi^{(e)}_{\ell m}\big) \;=\; -\, a_{0\ell m} \quad , \tag{4.46}$$

whence $\psi^{(e)}_{1\ell m}$ also obeys the same decoupled wave equation (4.36,39) as for $a_{2\ell m}$ and for $f^{(e)}_{\ell m}$, namely,

$$(\partial_{r*})^2\,\psi^{(e)}_{1\ell m} - (\partial_t)^2\,\psi^{(e)}_{1\ell m} - V_{1\ell}\,\psi^{(e)}_{1\ell m} \;=\; 0 \quad . \tag{4.47}$$

The spin-1 radial equation (4.36,39,47) in a Schwarzschild background, both for odd- and even-parity Maxwell fields, was first given in 1962 by Wheeler [66].

This suggests a 'preferred route' for understanding the even-parity perturbations (which are more complicated than in the odd-parity case, which only involves the single function $a_{2\ell m}(t,r)$, obeying the decoupled field equation

(4.36)): Given suitable boundary conditions, one first solves the linear decoupled wave equation in two variables t and r , namely, Eq.(4.47), for $\psi^{(e)}_{1\ell m}(t\,,r)$. By differentiation, following Eqs.(4.44,46), one then finds $a_{0\ell m}(t\,,r)$. Then, by integrating Eq.(4.41), one obtains also $a_{1\ell m}(t\,,r)$, and hence, from Eq.(4.40), $f^{(e)}_{\ell m}(t\,,r)$. From Eqs.(4.23-28), one now has all the non-zero components of the Maxwell field strength $F_{\mu\nu}$ in this even-parity case.

5 Regge-Wheeler formalism – odd-parity gravitational perturbations

Our boundary-value problem, as posed in the Introduction and in [11,16], involves specifying on the final space-like hypersurface Σ_F the spatial components $h^{(1)}_{ij}(x)$ $(i,j=1\,,2\,,3)$, of the real perturbations $h^{(1)}_{\mu\nu}(x)$ of the 4-metric $(\mu\,,\nu\,=\,0\,,1\,,2\,,3)$. We shall construct a basis of tensor spherical harmonics with which to expand the angular dependence of $h^{(1)}_{ij}$. In general, we make a multipole decomposition for real metric perturbations, of the form:

$$h^{(1)}_{ij}(x) \;=\; \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\Big[\big(h^{(-)}_{ij}\big)_{\ell m}(x)+\big(h^{(+)}_{ij}\big)_{\ell m}(x)\Big] \quad , \tag{5.1}$$

where $-$ and $+$ denote odd- and even-parity contributions, respectively. We comment below on the limit $\ell=2$ in the summation over ℓ .

In Sec.4, $s\,=\,1$ (Maxwell) perturbations of spherically-symmetric gravitational backgrounds were treated in the Regge-Wheeler (RW) formalism [63], and split naturally into odd and even type, according to their behaviour under parity inversion: $\theta\to\;(\pi-\theta)\,,\;\phi\to\;(\pi+\phi)$. The even 'electric-type' perturbations have parity $\pi=(-1)^{\ell}$, while the odd 'magnetic-type' perturbations have parity $\pi=(-1)^{\ell+1}$. The analogous (orthogonal) decomposition also holds for the $s=2$ gravitational-wave perturbations.

In the $s\,=\,1$ Maxwell case, the lowest $\ell\,=\,0$ mode does not propagate: in the electric case, it corresponds to the addition of a small charge to the black hole, to turn a Schwarzschild solution into a Reissner-Nordström solution [49] with charge $Q\ll M$. Correspondingly, in the $s\,=\,2$ case of gravitational perturbations, the multipoles with $\ell\,<\,2$ are non-radiatable. For example, the even-parity gravitational perturbations with $\ell\,=\,0$ correspond to a small static charge in the mass, while the $\ell=0$ odd-parity perturbation is identically zero. For $\ell\,=\,1$, the odd-parity (dipole) gravitational perturbations must be stationary [39], and even-parity dipole perturbations correspond to a coordinate displacement of the origin [67] and can be removed by a gauge transformation. For a general spin $s\,=\,0\,,1\,,2\,,$ perturbations with $\ell\,<\,|s|$ relate to total conserved quantities in the system. In the present $s\,=\,2$ gravitational-wave case, we consider accordingly only the propagating $\ell\,=\,2$ (quadrupole) and higher-ℓ modes.

In this Section and in Sec.7 below, we restrict attention to odd-parity gravitational-wave perturbations. Following Moncrief [68], we write

$$\left(h_{ij}^{(-)}\right)_{\ell m}(x) \;=\; h_{1\ell m}^{(-)}(t\,,r)\Big[(e_1)_{ij}\Big]_{\ell m} + h_{2\ell m}^{(-)}(t\,,r)\Big[(e_2)_{ij}\Big]_{\ell m} \quad . \tag{5.2}$$

(N.B. : one should not confuse the subscripts $1\,,2$ here with spin subscripts.) The non-zero components of the symmetric tensor fields $\big[(e_{1,2})_{ij}\big]_{\ell m}$ are defined by

$$\big[(e_1)_{r\theta}\big]_{\ell m} \;=\; -\,\big(\partial_\phi Y_{\ell m}\big)/\big(\sin\theta\big) \quad , \tag{5.3}$$

$$\big[(e_1)_{r\phi}\big]_{\ell m} \;=\; \big(\sin\theta\big)\big(\partial_\theta Y_{\ell m}\big) \quad , \tag{5.4}$$

$$\big[(e_2)_{\theta\theta}\big]_{\ell m} \;=\; \big(\sin\theta\big)^{-2}\Big[\big(\sin\theta\big)\partial^2_{\theta\phi} - \big(\cos\theta\big)\,\partial_\phi\Big]Y_{\ell m} \quad , \tag{5.5}$$

$$\big[(e_2)_{\theta\phi}\big]_{\ell m} \;=\; \frac{1}{2}\Big[\Big(\sin\theta\Big)^{-1}\partial^2_\phi - \big(\cos\theta\big)\,\partial_\theta - \big(\sin\theta\big)\,\partial^2_\theta\Big]Y_{\ell m} \quad , \tag{5.6}$$

$$\big[(e_2)_{\phi\phi})\big]_{\ell m} \;=\; \Big[\big(\cos\theta\big)\,\partial_\theta - \big(\sin\theta\big)\partial^2_{\theta\phi}\Big]Y_{\ell m} \quad . \tag{5.7}$$

This basis is normalised according to:

$$\int d\Omega\,\big[(e_1)^{ij}\big]_{\ell m}\,\big[(e_2)_{ij}\big]^*_{\ell' m'} \;=\; 0 \quad , \tag{5.8}$$

$$\int d\Omega\,\big[(e_1)^{ij}\big]_{\ell m}\,\big[(e_1)_{ij}\big]^*_{\ell' m'} \;=\; \frac{2e^{-a}}{r^2}\,\ell(\ell+1)\,\delta_{\ell\ell'}\,\delta_{mm'} \quad , \tag{5.9}$$

$$\int d\Omega\,\big[(e_2)^{ij}\big]_{\ell m}\,\big[(e_2)_{ij}\big]^*_{\ell' m'} \;=\; \frac{\ell(\ell+1)(\ell+2)(\ell-1)}{2r^4}\,\delta_{\ell\ell'}\,\delta_{mm'}\,, \tag{5.10}$$

where $\int d\Omega\,(\quad) \;=\; \int_0^{2\pi} d\phi \int_0^\pi d\theta\;\sin\theta\,(\quad)$, and where these indices are raised and lowered using the background 3-metric $\gamma^{ij}\,,\gamma_{ij}$. Note that both $\big[(e_1)_{ij}\big]_{\ell m}$ and $\big[(e_2)_{ij}\big]_{\ell m}$ are traceless.

In the standard 3 + 1 decomposition for the gravitational field [49], the 4-metric $g_{\mu\nu}$ is decomposed into the spatial 3-metric $h_{ij} \;=\; g_{ij}$ on a hypersurface $\{x^0 = \text{const.}\}$, together with the lapse function N and the shift vector field N^i , such that the 4-dimensional space-time metric has the form [15,49]:

$$ds^2 \;=\; h_{ij}\,(dx^i + N^i\,dt)\,(dx^j + N^j\,dt) \;-\; N^2 dt^2 \quad . \tag{5.11}$$

For odd-parity perturbations of the lapse, one has

$$N^{(1)(-)} \;=\; 0 \quad , \tag{5.12}$$

while the odd-parity shift vector takes the form

$$\Big[N_i^{\;(-)}\Big]_{\ell m} \;=\; h_{0\ell m}^{(-)}(t\,,r)\left[0\,, -\frac{1}{(\sin\theta)}\big(\partial_\phi Y_{\ell m}\big)\,,\,\big(\sin\theta\big)\big(\partial_\theta Y_{\ell m}\big)\right]. \tag{5.13}$$

For a real 4-metric $g_{\mu\nu}$, both $h_{ij}^{(1)}$ and $N_i^{(1)(-)}$ are real, and one has

$$h_{0,1,2\ell m}^{(-)*} \;=\; (-1)^m\,h_{0,1,2\ell,-m}^{(-)} \quad . \tag{5.14}$$

In the Hamiltonian formulation of general relativity, the momentum π^{ij} conjugate to the 'coordinate' h_{ij} is a symmetric spatial tensor density [49]. As with the 3-metric h_{ij} above [Eq.(5.1)], the linearised perturbations of π_{ij} can be decomposed into multipoles with odd or even parity:

$$\pi_{ij}^{(1)}(x) \;=\; \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\Big[\big(\pi_{ij}^{(-)}\big)_{\ell m}(x)+\big(\pi_{ij}^{(+)}\big)_{\ell m}(x)\Big] \quad . \tag{5.15}$$

Restricting attention at present to the odd modes, one has

$$\big(\pi_{ij}^{(-)}\big)_{\ell m} \;=\; ({}^{(3)}\gamma)^{\frac{1}{2}}\Big(p_{1\ell m}(t\,,r)\big[(e_1)_{ij}\big]_{\ell m}+p_{2\ell m}(t\,,r)\big[(e_2)_{ij}\big]_{\ell m}\Big)\,, \tag{5.16}$$

where $\big[(e_1)_{ij}\big]_{\ell m}$ and $\big[(e_2)_{ij}\big]_{\ell m}$ are given above. One finds that

$$p_{1\ell m}(t\,,r) \;=\; \frac{1}{2N^{(0)}}\left(\Big(\partial_t h_{1\ell m}^{(-)}\Big)-r^2\,\partial_r\left(\frac{h_{0\ell m}^{(-)}}{r^2}\right)\right) \quad , \tag{5.17}$$

$$p_{2\ell m}(t\,,r) \;=\; \frac{1}{2N^{(0)}}\left(\Big(\partial_t h_{2\ell m}^{(-)}\Big)+2\,h_{0\ell m}^{(-)}\right) \quad . \tag{5.18}$$

One can typically simplify the form of the perturbations by a gauge transformation (linearised coordinate transformation) in a neighbourhood of the final space-like hypersurface Σ_F. Suppose that the infinitesimal transformation is along a vector field ξ^μ. Then the metric perturbations transform infinitesimally by

$$g_{\mu\nu} \quad\rightarrow\quad g_{\mu\nu}-\nabla_\mu\xi_\nu-\nabla_\nu\xi_\mu \quad . \tag{5.19}$$

For odd perturbations, in the notation of Eq.(4.2), consider the infinitesimal vector field $\xi^{(-)\mu}$, with components [63] given by

$$\begin{aligned}\Big(\xi^{(-)t}\Big)_{\ell m} &= 0 \quad , \qquad \Big(\xi^{(-)r}\Big)_{\ell m} = 0 \quad ,\\ \Big(\xi^{(-)a}\Big)_{\ell m} &= \frac{\Lambda_{\ell m}(t\,,r)}{r^2}\,(\Phi_{\ell m})^a \quad .\end{aligned} \tag{5.20}$$

The resulting 'gauge transformation' is summarised by

$$h_{0\ell m}^{(-)'} \;=\; h_{0\ell m}^{(-)}-\big(\partial_t\Lambda_{\ell m}\big) \quad , \tag{5.21}$$

$$h_{1\ell m}^{(-)'} \;=\; h_{1\ell m}^{(-)}-\big(\partial_r\Lambda_{\ell m}\big)+\frac{2\Lambda_{\ell m}}{r} \quad , \tag{5.22}$$

$$h_{2\ell m}^{(-)'} \;=\; h_{2\ell m}^{(-)}+2\Lambda_{\ell m} \quad . \tag{5.23}$$

We have here neglected time-derivatives of the metric components: we are assuming that an Ansatz for the gauge functions $\Lambda_{\ell m}(t\,,r)$ based on separation of variables will be valid, involving frequencies which satisfy the adiabatic approximation [11,16] described below. In the Regge-Wheeler gauge [63], we set $h_{0\ell m}^{(-)RW} = h_{0\ell m}^{(-)'}$ and $h_{1\ell m}^{(-)RW} = h_{1\ell m}^{(-)'}$, as in Eqs.(5.21,22), but require also

$$h_{2\ell m}^{(-)RW} \;=\; 0 \;=\; h_{2\ell m}^{(-)}+2\,\Lambda_{\ell m} \quad . \tag{5.24}$$

For each ℓ, one can obtain solutions for arbitrary m by rotation from the case $m = 0$. Note also that the above equations show how the RW perturbations can be uniquely recovered from the perturbations in an arbitrary gauge.

Since odd-and even-parity perturbations decouple, the odd-parity field equations in the RW gauge are obtained by substituting Eq.(2.7), together with the equation $N_i^{(-)} = h_{ti}^{(-)}$ for the linearised shift vector, into the source-free linearised Einstein field equations [49]. The (Lorentzian) spherically-symmetric background metric is taken, as above, in the form of Eq.(4.10), and the mass function $m(t\,,r)$ is defined by Eq.(4.11). Then the odd-parity linearised field equations, taking respectively the $(t\phi)$, $(r\phi)$ and $(\theta\phi)$ components, read:

$$\big(\partial_r\big)^2 h_{0\ell m}^{(-)RW} - \partial_t\partial_r h_{1\ell m}^{(-)RW} - \frac{2}{r}\,\partial_t h_{1\ell m}^{(-)RW} + F_{1\ell}(t\,,r)\,h_{0\ell m}^{(-)RW} \;=\; 0\,, \tag{5.25}$$

and

$$\begin{aligned}
&\big(\partial_t\big)^2 h_{1\ell m}^{(-)RW} - \partial_t\partial_r h_{0\ell m}^{(-)RW} + \frac{2}{r}\,\partial_t h_{0\ell m}^{(-)RW} \\
&+ \frac{1}{2}\Big(\dot a + \dot b\Big)\Big(\partial_r h_{0\ell m}^{(-)RW} - \partial_t h_{1\ell m}^{(-)RW}\Big) \\
&- \bigg(\frac{1}{r}\Big(\dot a + \dot b\Big) + \frac{1}{2}\,b'\,\Big(\dot a - \dot b\Big)\bigg) h_{0\ell m}^{(-)RW} - F_{2\ell}(t\,,r)\,h_{1\ell m}^{(-)RW} \;=\; 0
\end{aligned}\quad , \tag{5.26}$$

and

$$\partial_t\Big(e^{(a-b)/2}\,h_{0\ell m}^{(-)RW}\Big) - \partial_r\Big(e^{(b-a)/2}\,h_{1\ell m}^{(-)RW}\Big) \;=\; 0 \quad . \tag{5.27}$$

Here,

$$F_{1\ell}(t\,,r) \;=\; \frac{e^a}{r^2}\bigg(\frac{4m}{r} + 4m' - \ell(\ell+1)\bigg) + e^{a-b}\bigg(\ddot a + \frac{1}{2}\,\dot a\Big(\dot a - \dot b\Big)\bigg) + Z\,, \tag{5.28}$$

and

$$F_{2\ell}(t\,,r) \;=\; -\,\frac{e^b}{r^2}\,(\ell+2)\,(\ell-1) + Z\,e^{b-a} + \ddot a + \frac{1}{2}\,\dot a\big(\dot a - \dot b\big)\,, \tag{5.29}$$

with

$$Z \;=\; -\,\frac{2e^a}{r}\Bigg(m'' + \bigg(\frac{2m'e^a}{r^2}\bigg)\Big(m' + rm\Big)\Bigg)\,. \tag{5.30}$$

Further, the Einstein field equations imply

$$m' \;=\; 4\pi r^2\rho \quad . \tag{5.31}$$

Here, ρ is the total energy density of all the radiative fields; in the present case ρ has contributions from $s = 0$ (massless scalar), $s = 2$ (graviton) and, if the Lagrangian contains Maxwell or Yang-Mills fields, also $s = 1$. In the adiabatic limit, in which $m(t\,,r)$ varies extremely slowly, Eq.(5.31) provides part of the description of the approximate Vaidya metric for the region of space-time in

which the black hole is evaporating; a much fuller treatment of the Vaidya region is given in [69].

Now consider the lowest-order perturbative contribution to the classical action of our present coupled bosonic system, for which the non-zero background part consists of the spherically-symmetric gravitational and scalar fields, $(\gamma_{\mu\nu}\,,\,\Phi)$. It is assumed that the background fields obey the coupled Einstein/massless-scalar field equations and contribute $S^{(0)}_{\rm class}$ to the classical action. Any spin-1 fields present, whether Maxwell or Yang-Mills, propagate at lowest order in the background, and the spin-1 contribution to the classical action begins at second order, as described in Sec.4, and adds to the various lowest-order perturbative Einstein-Hilbert and scalar contributions. To simplify the exposition at this point, let us temporarily neglect the spin-1 field.

After detailed calculation [33,34], one finds that the classical Lorentzian action for an (anisotropic) Einstein/massless-scalar solution, subject to perturbed boundary data $\big(h^{(0)}_{ij}\,+\,h^{(1)}_{ij}\,,\,\Phi\,+\,\phi^{(1)}\big)$ on the initial and final surfaces Σ_I and Σ_F, can be written as the background contribution $S^{(0)}_{\rm class}$ above, plus a quadratic-order contribution, plus higher-order terms. Thus:

$$\begin{aligned} S_{\rm class} \;&=\; S^{(0)}_{\rm class} + S^{(2)}_{\rm class} + \;\ldots \quad , \\ &=\; \frac{1}{32\pi}\int_{\Sigma_F} d^3x\,\Big(\pi^{(0)ij}\,h^{(0)}_{ij} + \pi^{(1)ij}\,h^{(1)}{}_{ij}\Big) + \;\ldots \\ &+\; \frac{1}{2}\int_{\Sigma_F} d^3x\,\Big(\Phi\,\Pi_\phi + \phi^{(1)}\,\pi_\phi{}^{(1)}\Big) \;+\;\ldots \\ &-\; M\,T \quad . \end{aligned} \tag{5.32}$$

Again, for simplicity, it is assumed that no perturbations are prescribed on the initial boundary Σ_I, but that there are non-zero perturbations on the final boundary Σ_F. As described after Eq.(5.14), π^{ij} is defined to be the momentum conjugate to the 'coordinate' h_{ij}. Similarly, π_ϕ is the momentum conjugate to the variable ϕ, and is given by the normal future-directed derivative $\partial\phi/\partial n$. In Eq.(5.32), T, as usual, denotes the Lorentzian proper-time interval between the initial and final boundaries, as measured at spatial infinity. Further, M denotes the mass of the space-time; for a well-posed (complexified) classical boundary-value problem, the mass defined on the initial surface must agree with the mass defined on the final surface.

We can now compute the odd-parity contribution to the classical gravitational action. In an arbitrary gauge, taking only the intrinsic metric perturbation $h^{(1)}_{ij}$ to be non-zero on Σ_F, but $\phi^{(1)}=0$ there, one has the second-variation part of the action

$$S^{(2)}_{\rm class}\Big[h^{(1)}_{ij}\Big] \;=\; \frac{1}{32\pi}\int_{\Sigma_F} d^3x\;\pi^{(1)ij}\;h^{(1)}_{ij} \quad . \tag{5.33}$$

On discarding a total divergence, the spin-2 classical action can also be written

as

$$
\begin{aligned}
S^{(2)}_{\text{class}} &= \frac{1}{64\pi}\int_{\Sigma_F} d^3x\,\sqrt{{}^{(3)}\gamma}\;n^{(0)\mu}\left(\bar{h}^{(1)\mu\nu}\,\nabla_\alpha h^{(1)}_{\mu\nu}\,-2\,h^{(1)}_{\alpha\nu}\,\nabla_\rho h^{(1)\nu\rho}\right)\\
&\quad+\frac{1}{16\pi}\int_{\Sigma_F} d^3x\,\sqrt{{}^{(3)}\gamma}\left(\frac{N^{(1)}{}_i}{N^{(0)}}\right)\bar{h}^{(1)ik}{}_{|k}\quad .
\end{aligned}
\tag{5.34}
$$

Since odd- and even-parity perturbations are orthogonal, there are no cross-terms in the action. In an arbitrary gauge, the odd-parity contribution to the classical gravitational action can be re-written as

$$
\begin{aligned}
S^{(2)}_{\text{class}}\Big[(h^{(-)}_{ij})_{\ell m}\Big] &= \frac{1}{32\pi}\int_{\Sigma_F} d^3x \sum_{\ell\ell' mm'}\left(\pi^{(-)ij}\right)_{\ell m}\left(h^{(-)}_{ij}\right)^{*}_{\ell' m'}\\
= \frac{1}{32\pi}\sum_{\ell m}\ell(\ell+1)&\int_0^{R\infty} dr\;h^{(-)*}_{1\ell m}\left(\left(\partial_t h^{(-)}_{1\ell m}\right)+\frac{2}{r}\,h^{(-)}_{0\ell m}-\left(\partial_r h^{(-)}_{0\ell m}\right)\right)\Bigg|_T\\
+\;\frac{1}{128\pi}\sum_{\ell m}\frac{(\ell+2)!}{(\ell-2)!}&\int_0^{R_\infty} dr\;e^a\,\frac{h^{(-)*}_{2\ell m}}{r^2}\left(\left(\partial_t h^{(-)}_{2\ell m}\right)+2\,h^{(-)}_{0\ell m}\right)\Bigg|_T\;,
\end{aligned}
\tag{5.35}
$$

Here, we have used Eqs.(5.8-10,17,18), and have taken the perturbations to vanish initially. Note that, if we were to evaluate Eq.(5.35) in the RW gauge, for which $h^{(-)RW}_{2\ell m} = 0$, so that the second integral would vanish, and then substitute for the Regge-Wheeler functions *via* Eqs.(5.21-24), then, in the adiabatic approximation, we would arrive back at Eq.(5.35) up to a boundary term

$$
h^{(-)*}_{2\ell m}\;P^{(-)}_{\ell m}\Big|_{r=0}^{r=R_\infty}\quad , \tag{5.36}
$$

where

$$
P^{(-)}_{\ell m} \;=\; \ell(\ell+1)\left(\left(\partial_t h^{(-)}_{1\ell m}\right)-r^2\,\partial_r\left(\frac{h^{(-)}_{0\ell m}}{r^2}\right)\right). \tag{5.37}
$$

Thus, $S^{(2)}_{\text{class}}\big[(h^{(-)}_{ij})_{\ell m}\big]$ is gauge-invariant up to a boundary term. We shall return below to the question of boundary conditions for the odd-parity perturbations.

At first sight, the odd-parity action looks unwieldy. Ideally, we would like to work with a classical action (both for odd and for even parity) in the form $\int dr\,\psi(\partial_t\psi)$, of the same general kind as in the massless-scalar classical action in [33,34]. In the present gravitational case, ψ would ideally also be gauge-invariant and would obey a wave equation with a real potential. To achieve this form, first use Eqs.(5.21-24) for the RW functions, and substitute them into the field equation (5.26), to obtain

$$
\left(\partial_t P^{(-)}_{\ell m}\right) \;=\; \ell\,(\ell+1)F_{2\ell}(r)\left(h^{(-)}_{1\ell m}+\frac{1}{2}\,\partial_r\left(\frac{h^{(-)}_{2\ell m}}{r^2}\right)\right), \tag{5.38}
$$

$$\left(\partial_r P^{(-)}_{\ell m}\right) = -\frac{2P^{(-)}_{\ell m}}{r} + \ell(\ell+1)\left(\frac{1}{2}F_{1\ell}(r) + \frac{1}{r^2}\right)\left(\left(\partial_t h^{(-)}_{2\ell m}\right) + 2h^{(-)}_{0\ell m}\right) \quad (5.39)$$

When we substitute into Eq.(5.35), using Eq.(5.38) for $h^{(-)}_{1\ell m}$, and then use Eqs.(5.29,39), the boundary term (5.36) vanishes. As a consequence, we find in the adiabatic approximation that

$$S^{(2)}_{\mathrm{class}}\left[\left(h^{(-)}_{ij}\right)_{\ell m}\right] = -\frac{1}{32\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(\ell-2)!}{(\ell+2)!}\int_0^{R_\infty} dr\, e^{a}\,\xi^{(-)}_{2\ell m}\left(\partial_t \xi^{(-)*}_{2\ell m}\right)\Bigg|_{t=T}\,, \tag{5.40}$$

where $\xi^{(-)}_{2\ell m}$ is defined by

$$\xi^{(-)}_{2\ell m} \;=\; r\,P^{(-)}_{\ell m}\,. \tag{5.41}$$

Eqs.(5.21-23) show that $\xi^{(-)}_{2\ell m}$ is gauge-invariant. Indeed, $\xi^{(-)}_{2\ell m}$ is related to Moncrief's [68] gauge-invariant generalisation of the Zerilli function, $Q^{(-)}_{\ell m}$ [64,70], defined as

$$Q^{(-)}_{\ell m} \;=\; \frac{e^{-a}}{r}\left(h^{(-)}_{1\ell m} + \frac{1}{2}\,r^2\,\partial_r\left(\frac{h^{(-)}_{2\ell m}}{r^2}\right)\right)\,, \tag{5.42}$$

by

$$Q^{(-)}_{\ell m} \;=\; -\;\frac{(\ell-2)!}{(\ell+2)!}\left(\partial_t \xi^{(-)}_{2\ell m}\right)\,, \tag{5.43}$$

which, in effect, replicates Eq.(5.38). Note the simplifying property of $Q^{(-)}_{\ell m}$, namely, that it is written entirely in terms of perturbations of the 3-geometry (our chosen boundary data). Further, $Q^{(-)}_{\ell m}$ is automatically gauge-invariant, since it is independent of the perturbed lapse and shift.

In the adiabatic approximation, the function $\xi^{(-)}_{\ell m}$ obeys the wave equation, of RW type:

$$e^{-a}\,\partial_r\left(e^{-a}\left(\partial_r \xi^{(-)}_{2\ell m}\right)\right) - \left(\partial_t\right)^2\xi^{(-)}_{2\ell m} - V^{(-)}_{\ell}(r)\,\xi^{(-)}_{\ell m} \;=\; 0\,, \tag{5.44}$$

where

$$V^{(-)}_{\ell}(r) \;=\; e^{-a}\left(\frac{\ell(\ell+1)}{r^2} - \frac{6m(r)}{r^3}\right) \quad > \quad 0\,. \tag{5.45}$$

Further, in the adiabatic approximation, the function $Q^{(-)}_{\ell m}$ obeys the same equation (5.44). In the RW gauge, one would solve for $Q^{(-)RW}_{\ell m}$, then determine $h^{(-)RW}_{1\ell m}$ from Eq.(5.42), and then determine $h^{(-)RW}_{0\ell m}$ with the help of Eq.(5.27).

6 Boundary conditions and classical action – Maxwell case

In Sec.4, for the Maxwell field (whether odd- or even-parity), we derived an expression, Eq.(4.45), for the classical Maxwell action $S^{\mathrm{EM}}_{\mathrm{class}}$, given *via* certain

operations in terms of the magnetic field on the final boundary Σ_F . In Sec.5, for odd-parity gravitational perturbations, we derived Eq.(5.40), a somewhat analogous expression for the second-variation classical action $S^{(2)}_{\rm class}$, depending on the odd-parity metric perturbations $h^{(-)}_{ij}$ over the final surface Σ_F . In the present Section 6, concerning Maxwell theory, we relate $S^{\rm EM}_{\rm class}$ more explicitly to the final boundary data. The following Sec.7 will similarly re-express $S^{(2)}_{\rm class}$ for odd gravitational perturbations. A corresponding treatment for the more complicated even-parity gravitational case will be given in Secs.8,9. The resulting expressions for the classical action will greatly simplify our understanding of the black-hole quantum amplitudes.

Physically, our gauge-invariant odd- and even-parity Maxwell variables $\psi^{(o)}_{1\ell m}$, $\psi^{(e)}_{1\ell m}$ are effectively the radial components of the magnetic and electric field strengths, respectively:

$$B^{(1)r}_{\ell m}(x) \;=\; \frac{e^{-a/2}}{r^2}\,\psi^{(o)}_{1\ell m}(t\,,r)\,Y_{\ell m}(\Omega) \quad , \tag{6.1}$$

$$E^{(1)r}_{\ell m}(x) \;=\; -\,\frac{e^{-a/2}}{r^2}\,\psi^{(e)}_{1\ell m}(t\,,r)\,Y_{\ell m}(\Omega) \quad , \tag{6.2}$$

where

$$B^{(1)i}_{\ell m} \;=\; \big({}^{(3)}\gamma\big)^{-\frac{1}{2}}\,\mathcal{B}^{(1)i}_{\ell m} \quad . \tag{6.3}$$

The remaining, transverse, magnetic field components are

$$B^{(1)\theta}_{\ell m}(x) \;=\; \frac{e^{-a/2}}{r^2}\left[a_{1\ell m}(t,r)\frac{\big(\partial_\phi Y_{\ell m}\big)}{\big(\sin\theta\big)} + \frac{\big(\partial_r\psi^{(o)}_{1\ell m}\big)}{\ell(\ell+1)}\big(\partial_\theta Y_{\ell m}\big)\right] . \tag{6.4}$$

$$B^{(1)\phi}_{\ell m}(x) \;=\; \frac{e^{-a/2}}{r^2\big(\sin\theta\big)}\left[-\,a_{1\ell m}(t,r)\big(\partial_\theta Y_{\ell m}\big) + \frac{\big(\partial_r\psi^{(o)}_{1\ell m}\big)}{\ell(\ell+1)}\frac{\big(\partial_\phi Y_{\ell m}\big)}{\big(\sin\theta\big)}\right] . \tag{6.5}$$

The main aim of this paper is to calculate quantum amplitudes for weak spin-1 (Maxwell) perturbative data on the late-time final surface Σ_F , by evaluating the classical action $S^{EM}_{\rm class}$ and hence the semi-classical wave function (const.) $\times\exp\big(iS^{EM}_{\rm class}\big)$, *as functionals of the spin-1 final boundary data.* In Eq.(4.45), $S^{EM}_{\rm class}$ was expressed as an integral over the boundary, involving various perturbative quantities used in the description above of the dynamical perturbations. The present task is to determine 'optimal' or 'natural' boundary data, both for the odd-parity case and separately for the even-parity case, such that (i.) the classical boundary-value problem can readily be solved, given these data, and (ii.) the classical Maxwell action $S^{EM}_{\rm class}$ can be (re-)expressed in terms of the appropriate boundary data. Under those conditions, we will then have a description of the spin-1 radiation, associated with gravitational collapse to a black hole, analogous to that for the spin-0 (massless-scalar) radiation, developed in [33,34].

Following the discussion of Secs.2 and 4, the relevant field components to be fixed on Σ_I and Σ_F, in the odd-parity case, are $\psi^{(o)}_{1\ell m}$, as may be seen

from Eq.(6.1). In the even-parity case, the appropriate boundary conditions involve fixing $a_{1\ell m}$, as may be seen from Eqs.(6.4,5). In the even case, this is equivalent, from Eqs.(4.40,42), to specifying $\partial_t\psi^{(e)}_{1\ell m}$ on the boundary. Thus, we choose the boundary data to consist of $\psi^{(o)}_{1\ell m}$ in the odd-parity case, and $\partial_t\psi^{(e)}_{1\ell m}$ in the even-parity case. Hence, though $\psi^{(o)}_{1\ell m}$ and $\psi^{(e)}_{1\ell m}$ both obey the same dynamical field equations (4.36,43) and (4.47), the natural boundary conditions for them on $\Sigma_{I,F}$ are quite different – Dirichlet for the odd-parity case $\psi^{(o)}_{1\ell m}$, but Neumann in the even-parity case $\psi^{(e)}_{1\ell m}$. This is reminiscent of the situation obtaining when spin-2 gravity is coupled to all lower spins, fermionic as well as bosonic; that is, to spins $s=\frac{3}{2}\,,\,1\,,\,\frac{1}{2}$ and 0 , especially in locally-supersymmetric models [22], such as models of $N=1$ supergravity with gauged supermatter [71-73]. As found also, in this paper, for spin-2 (graviton) perturbed data on the final surface Σ_F , the natural boundary conditions are contrasting, for odd-parity $vis-\grave{a}-vis$ even-parity modes. For the remaining $s=0$ (scalar) bosonic case, if one requires, as in the Introduction, that the full theory be locally supersymmetric, so that quantum amplitudes are meaningful, one finds that all scalar fields must be *complex* , whether as part of a multiplet or as a single complex scalar [22]. The treatment of the real $s=0$ case in [33,34] can be replicated in the case of a complex scalar field ϕ , except that the natural boundary conditions, consistent with local supersymmetry, require $\mathrm{Re}(\phi)$ to be fixed at a surface such as Σ_F (Dirichlet), whereas the normal derivative $\partial[\mathrm{Im}(\phi)]/\partial n$ must also be fixed at a bounding surface (Neumann). Of course, this treatment extends to fermionic data ($s=\frac{1}{2}$ and $\frac{3}{2}$) , as described in [35,36].

In the gravitational-collapse model, by analogy with the simplifying choice $\phi^{(1)}|_{\Sigma_I}=0$ for the initial perturbative scalar-field data, taken in [33,34], we take (for the purposes of exposition) the simplest Maxwell initial data at Σ_I $(t=0)$. That is, we consider a negligibly weak magnetic field outside the 'star': the boundary conditions are

$$\psi^{(o)}_{1\ell m}(0\,,r) \;=\; 0 \quad , \tag{6.6}$$
$$\partial_t\psi^{(e)}_{1\ell m}(0\,,r) \;=\; 0 \quad . \tag{6.7}$$

Condition (6.6) is a Dirichlet condition on the initial odd-parity magnetic field – see Eqs.(6.1,4,5). Condition (6.7) implies that we have an initially static even-parity multipole [65].

We now follow the analysis of the spin-0 field, and separate the radial- and time-dependence. In neighbourhoods of Σ_I and Σ_F , where an adiabatic approximation is valid, we can 'Fourier-expand' the variables $\psi^{(o)}_{1\ell m}(t\,,r)$ and $\psi^{(e)}_{1\ell m}(t\,,r)$, subject to the initial conditions (6.6) and (6.7). By analogy with the scalar case [33,34], let us write

$$\psi^{(o)}_{1\ell m}(t\,,r) \;=\; \int_{-\infty}^{\infty} dk\; a^{(o)}_{1k\ell m}\,\psi^{(o)}_{1k\ell}(r)\,\frac{\sin(kt)}{\sin(kT)} \quad , \tag{6.8}$$
$$\psi^{(e)}_{1\ell m}(t,r) \;=\; \int_{-\infty}^{\infty} dk\; a^{(e)}_{1k\ell m}\,\psi^{(e)}_{1k\ell}(r)\,\frac{\cos(kt)}{\sin(kT)} \quad , \tag{6.9}$$

where the radial functions $\{\psi^{(o)}_{1k\ell}(r)\}$ and $\{\psi^{(e)}_{1k\ell}(r)\}$ are independent of m, given the spherical symmetry of the background space-time. Here, the position-independent quantities $\{a^{(o)}_{1k\ell m}\}$ and $\{a^{(e)}_{1k\ell m}\}$ are some coefficients, with smooth dependence on the continuous variable k, which label the configuration of the electromagnetic field on the final surface Σ_F.

The radial functions $\{\psi^{(o)}_{1k\ell}(r)\}$ and $\{\psi^{(e)}_{1k\ell}(r)\}$ each obey a regularity condition at the centre of spherical symmetry $\{r=0\}$ on the final surface Σ_F; this requires that the corresponding (spatial) electric or magnetic field, defined *via* Eqs.(6.1-5), should be smooth in a neighbourhood of $\{r=0\}$. As a consequence, the radial functions must be real:

$$\psi^{(o)\,*}_{1k\ell}(r) \;=\; \psi^{(o)}_{1,-k\ell}(r) \quad , \qquad \psi^{(e)\,*}_{1k\ell}(r) \;=\; \psi^{(e)}_{1,-k\ell}(r) \quad . \tag{6.10}$$

For small r, the radial functions should be asymptotically proportional to a spherical Bessel function [74]:

$$\psi^{(o)}_{1k\ell}(r) \quad \sim \quad (\text{const.}) \times \; r\, j_\ell(kr) \quad , \tag{6.11}$$
$$\psi^{(e)}_{1k\ell}(r) \quad \sim \quad (\text{const.})' \times \; r\, j_\ell(kr) \quad , \tag{6.12}$$

as $r \to 0_+$. Also, the reality of the radial electric and magnetic fields implies that

$$\psi^{(o)}_{1\ell m}(t\,,r) \;=\; (-1)^m\, \psi^{(o)*}_{1\ell,-m}(t\,,r)\,, \qquad \psi^{(e)}_{1\ell m}(t\,,r) \;=\; (-1)^m\, \psi^{(e)*}_{1\ell,-m}(t\,,r)\,. \tag{6.13}$$

This in turn implies that

$$a^{(o)}_{1k\ell m} \;=\; (-1)^m\, a^{(o)*}_{1,-k\ell,-m} \;, \qquad a^{(e)}_{1k\ell m} \;=\; (-1)^{m+1}\, a^{(e)*}_{1,-k\ell,-m} \;. \tag{6.14}$$

Since the potential (4.37), appearing in the $(t\,,r)$ wave equation (4.36), tends sufficiently rapidly to zero as $r \to \infty$, in the region where the space-time is almost Schwarzschild, one has asymptotic (large-r) behaviour of $\psi^{(o)}_{1k\ell}(r)$ and $\psi^{(e)}_{1k\ell}(r)$ which is analogous to that in the scalar case:

$$\psi^{(o)}_{1k\ell}(r) \quad \sim \quad \Big(z^{(o)}_{k\ell}\, \exp(ik\, r^*_s) + z^{(o)*}_{k\ell}\, \exp\big(-ik\, r^*_s\big)\Big)\,. \tag{6.15}$$
$$\psi^{(e)}_{1k\ell}(r) \quad \sim \quad \Big(z^{(e)}_{k\ell}\, \exp(ik\, r^*_s) + z^{(e)*}_{k\ell}\, \exp\big(-ik\, r^*_s\big)\Big)\,. \tag{6.16}$$

Here, $\{z^{(o)}_{k\ell}\}$ and $\{z^{(e)}_{k\ell}\}$ are complex coefficients, depending smoothly on the continuous variable k. Also, r^*_s again denotes the Regge-Wheeler 'tortoise' coordinate of Eq.(4.38) for the Schwarzschild geometry [49,63]. As in the scalar case [33,34], the inner product (normalisation) for the radial functions follows in the limit $R_\infty \to \infty$:

$$\int_0^{R_\infty} dr\; e^a\, \psi^{(o)}_{1k\ell}(r)\, \psi^{(o)}_{1k'\ell}(r) \;=\; 2\pi\, |z^{(o)}_{k\ell}|^2 \Big[\delta(k\,,k') + \delta(k\,,-k')\Big]\,, \tag{6.17}$$
$$\int_0^{R_\infty} dr\; e^a\, \psi^{(e)}_{1k\ell}(r)\, \psi^{(e)}_{1k'\ell}(r) \;=\; 2\pi\, |z^{(e)}_{k\ell}|^2 \Big[\delta(k\,,k') + \delta(k\,,-k')\Big]\,. \tag{6.18}$$

Finally, we are in a position to compute the classical Maxwell action $S^{EM}_{\rm class}$ as a functional of the spin-1 boundary data on the final surface Σ_F . This gives straightforwardly the semi-classical wave function for the complexified time-interval T , leading to the Lorentzian quantum amplitude or wave function. Our boundary conditions (6.6,7) above on the initial hypersurface Σ_I , at time $t = 0$ say, were designed so as to give zero contribution from Σ_I to the expression (4.45) for the classical action $S^{EM}_{\rm class}$. The contribution to (4.45) from Σ_F is found, using Eqs.(6.8,9,17,18), to be

$$\begin{aligned} & S^{EM}_{\rm class}\big[\{a^{(o)}_{1k\ell m}\,,\,a^{(e)}_{1k\ell m}\}\big] \\ = & -\frac{1}{2}\sum_{\ell m}\frac{(\ell-1)!}{(\ell+1)!}\int_0^\infty dk\;k\,\Bigg[|z^{(o)}_{k\ell}|^2\left(|a^{(o)}_{1k\ell m}|^2+{\rm Re}\Big(a^{(o)}_{1k\ell m}\,a^{(o)*}_{1,-k\ell m}\Big)\right) \\ & +\,|z^{(e)}_{k\ell}|^2\left(|a^{(e)}_{1k\ell m}|^2+{\rm Re}\Big(a^{(e)}_{1k\ell m}\,a^{(e)*}_{1,-k\ell m}\Big)\right)\Bigg]\,\cot(kT)\big|_{\Sigma_F}\ . \end{aligned} \qquad (6.19)$$

As promised, this does now express the classical Maxwell part $S^{EM}_{\rm class}$ of the action as an explicit functional of suitably chosen boundary data, namely, $\{a^{(o)}_{1k\ell m}\}$ and $\{a^{(e)}_{1k\ell m}\}$.

7 Boundary conditions and asymptotically-flat gauge – odd-parity gravitational perturbations

In classical Lorentzian general relativity, one would expect to choose regular Cauchy data on an initial space-like hypersurface Σ_I , which would then evolve smoothly into $\{x^0 > 0\}$, subject to the linear hyperbolic equation (5.44). A natural initial condition, for given quantum numbers $(\ell\,, m)$ [65], would be to assume an initially stationary odd-parity multipole:

$$\Big(\partial_t \xi^{(-)}_{2\ell m}\Big)\Big|_{t=0} \;=\; 0 \quad . \qquad (7.1)$$

The combined Einstein/massless-scalar boundary-value problem, originally posed in [33,34], for complex time-separation $T = |T|\,\exp(-\,i\theta)\ ,\ 0 < \theta \le \pi/2\,$, involved specifying the intrinsic 3-metric $(h_{ij})_{I,F}$ and the value of the scalar field $(\phi)_{I,F}$ on the initial and final space-like hypersurfaces $\Sigma_I\,,\Sigma_F$. By Eq.(5.43), the above Eq.(7.1) reads

$$Q^{(-)}_{\ell m}(0\,,r) \;=\; 0 \qquad (7.2)$$

or, equivalently,

$$h^{(-)}_{1\ell m}(0\,,r) \;=\; 0 \quad , \qquad (7.3)$$

$$h^{(-)}_{2\ell m}(0\,,r) \;=\; 0 \quad , \qquad (7.4)$$

We therefore take these as our (Dirichlet) boundary conditions on the odd-parity gravitational perturbations, on the initial surface Σ_I , even though they may have arisen from consideration of the Cauchy problem.

In [11,16,33,34] for the $s = 0$ case, we made use of the adiabatic approximation in order to separate the perturbation problem with respect to the variables t and r. Here, for $s = 2$, we first separate the odd-parity Eqs.(5.25,26) in the RW gauge, and then use Eqs.(5.21-23) to determine the time-dependence (in particular) in any gauge.

As in the massless-scalar ($s = 0$) case, we introduce a 'Fourier-type' expansion:

$$h^{(-)RW}_{0,1,2\ell m}(t\,,r) \;=\; \int_{-\infty}^{\infty} dk\; a^{(-)}_{k\ell m}\; h^{(-)RW}_{0,1,2k\ell m}(t\,,r) \quad , \tag{7.5}$$

where the $\{a^{(-)}_{k\ell m}\}$ are certain odd-parity 'Fourier'coefficients. With suitable treatment of any arbitrary phase factors involved, in order to separate the odd-parity field equations (5.22,23) in the adiabatic approximation, one must have

$$h^{(-)RW}_{0\ell m}(t\,,r) \quad \propto \quad \cos(kt) \quad , \tag{7.6}$$

$$h^{(-)RW}_{1\ell m}(t\,,r) \quad \propto \quad \sin(kt) \quad , \tag{7.7}$$

(Of course, a normal-mode e^{-ikt} time dependence for the functions $h^{(-)RW}_{0\,,1\,,\ell m}$ would also satisfy the field equations.) In Eq.(5.23), if $h^{(-)RW}_{0\ell m}$, which is related to the odd-parity shift and can thus be freely specified, has $\cos(kt)$ time dependence, then $h^{(-)}_{0\ell m}$ must have the same time-dependence, while $\Lambda_{\ell m}(t\,,r)$ must have $\sin(kt)$ time dependence. But, by Eq.(5.21), $h^{(-)}_{2\ell m}$ must then have $\sin(kt)$ time-dependence. Similarly, from Eq.(5.22), given that $h^{(-)RW}_{1\ell m}$ has $\sin(kt)$ time-dependence, $h^{(-)}_{1\ell m}$ must also have $\sin(kt)$ time-dependence, as must $\Lambda_{\ell m}(t\,,r)$. These conclusions are consistent with our choice of boundary conditions (7.3,4). Noting Eqs.(5.37,42,43), the Dirichlet conditions (7.3,4) are equivalent to the boundary condition (7.1), which is analogous to a specification of momenta in a $\left(\xi^{(-)}_{2\ell m}\,,\,\partial_t\xi^{(-)}_{2\ell m}\right)$ representation. This also accounts for the minus sign in Eq.(5.40).

For large r, the potential term in Eq.(5.44) vanishes sufficiently rapidly that $\xi^{(-)}_{2\ell m}$ becomes a superposition of outgoing and ingoing waves at radial infinity. Note that $Q^{(-)}_{\ell m}$ also obeys Eq.(5.44); thus, Eq.(5.42) in the RW gauge tells us that $h^{(-)RW}_{1\ell m} \;=\; r\,Q^{(-)RW}_{\ell m}\,e^a \;=\; O(r)$ at large r. Now, the field equation (5.27) implies that

$$\left(\partial_t h^{(-)RW}_{o\ell m}\right) \;=\; e^{-a}\,\partial_r\left(r\,Q^{(-)RW}_{\ell m}\right) \quad . \tag{7.8}$$

That is, odd-parity metric perturbations diverge at large r, in the RW gauge. This is only a coordinate effect, as the Riemann-curvature invariants decay at a rate $O(r^{-1})$ at large r [42,43]. (A similar phenomenon occurs for the even-parity perturbations in the RW gauge.) Here, in the odd-parity case, we construct a gauge transformation to an asymptotically-flat (AF) gauge, in which the radiative behaviour of the metric perturbations becomes manifest.

Our odd-parity AF gauge is chosen such that

$$h^{(-)AF}_{0\ell m}(t\,,r) \;=\; 0 \quad . \tag{7.9}$$

Thus, in terms of the preceding RW gauge:

$$h^{(-)AF}_{0\ell m} \;=\; 0 \;=\; h^{(-)RW}_{0\ell m} - \Big(\partial_t \Lambda_{\ell m}\Big) \quad , \tag{7.10}$$

$$h^{(-)AF}_{1\ell m} \;=\; h^{(-)RW}_{1\ell m} - \Big(\partial_r \Lambda_{\ell m}\Big) + \frac{2\Lambda_{\ell m}}{r} \quad , \tag{7.11}$$

$$h^{(-)AF}_{2\ell m} \;=\; 2\,\Lambda_{\ell m} \quad . \tag{7.12}$$

Given $h^{(-)RW}_{0\ell m}$ and $h^{(-)RW}_{1\ell m}$ as a starting-point, one can, from the above, determine $\Lambda_{\ell m}(t\,,r)$ and hence $h^{(-)AF}_{1\ell m}$ and $h^{(-)AF}_{2\ell m}$. On substituting for $h^{(-)RW}_{1\ell m}$ from Eq.(7.11) into Eq.(5.24), one finds

$$\Big(\partial_t\Big)^2 h^{(-)AF}_{1\ell m} \;=\; -\,\frac{2\lambda e^{-a}}{r^2}\,h^{(-)RW}_{1\ell m} \quad . \tag{7.13}$$

Now, following the approach used throughout when studying boundary conditions at the final surface Σ_F $(t=T)$, set:

$$h^{(-)AF}_{1\ell m}(t\,,r) \;=\; \int_{-\infty}^{\infty} dk\; a^{(-)}_{k\ell m}\; h^{(o)AF}_{1k\ell}(r)\,\frac{\sin(kt)}{\sin(kT)} \quad , \tag{7.14}$$

where the $\{h^{(-)AF}_{1k\ell}(r)\}$ are real radial functions.

The property $a^{(-)*}_{k\ell m} = (-1)^m a^{(-)}_{-k\ell,-m}$ holds for the coefficients. Similarly, one can construct a corresponding expansion for $Q^{(-)RW}_{\ell m}(t\,,r)$. Then, from Eq.(7.13), one has

$$h^{(-)AF}_{1\ell m}(t\,,r) \;=\; \frac{2\lambda}{r}\int_{-\infty}^{\infty} dk\;\frac{a^{(-)}_{k\ell m}}{k^2}\;Q^{(-)RW}_{k\ell}(r)\,\frac{\sin(kt)}{\sin(kT)} \quad , \tag{7.15}$$

which is $O(r^{-1})$ at large r, as required. On using Eq.(5.37), one further finds

$$\begin{aligned} \xi^{(-)AF}_{2\ell m}(t\,,r) \;&=\; r\ell(\ell+1)\Big(\partial_t h^{(-)AF}_{1\ell m}\Big) \\ &=\; \int_{-\infty}^{\infty} dk\;\hat{a}^{(-)}_{2k\ell m}\;\xi^{(-)AF}_{2k\ell}(r)\,\frac{\cos(kt)}{\sin(kT)} \quad , \end{aligned} \tag{7.16}$$

where

$$\hat{a}^{(-)}_{2k\ell m} \;=\; k\ell(\ell+1)a^{(-)}_{k\ell m} \quad , \tag{7.17}$$

and where

$$\xi^{(-)AF}_{2k\ell}(r) \;=\; r\,h^{(-)AF}_{1k\ell}(r) \tag{7.18}$$

satisfies

$$e^{-a}\Big(e^{-a}\,\xi^{(-)AF\prime}_{2k\ell}\Big)' + \Big(k^2 - V^{(-)}_{\ell}(r)\Big)\xi^{(-)AF}_{2k\ell} \;=\; 0 \quad . \tag{7.19}$$

As in the spin-0 case [33,34] and in the spin-1 case above, we have, for $k>0$:

$$\xi^{AF}_{2k\ell-}(r) \sim r\,j_\ell(kr) \;, \qquad \text{as} \quad r \to 0 \quad , \tag{7.20}$$

$$\xi^{AF}_{2k\ell-}(r) \sim \Big(\Big(z_{2k\ell-}\Big)\exp\big(ikr^*_s\big) + \Big(z^*_{2k\ell-}\Big)\exp\big(-ikr^*_s\big)\Big), \text{ as } r^*_s \to \infty \qquad (7.21)$$

where the $j_\ell(z)$ are spherical Bessel functions, and where r^*_s is the Schwarzschild Regge-Wheeler coordinate [49,63] of Eq.(4.38). Thence, one deduces the normalisation property

$$\int_0^{R\infty} dr\, e^a\, \xi^{AF}_{2k\ell-}(r)\, \xi^{AF}_{2k'\ell-}(r)\Big|_{\Sigma_F} = 2\pi\, |z_{2k\ell-}|^2 \Big(\delta(k\,,k') + \delta(k\,,-k')\Big)\,. \quad (7.22)$$

The resulting form of the classical action for odd-parity (spin-2) gravitational perturbations can then be expressed as a functional of the complex quantities $\{a_{2k\ell m-}\}$ which encode the boundary data on Σ_F for the odd-parity gravitational perturbations. Here,

$$S^{(2)}_{\rm class}\Big[\{a_{2k\ell m-}\}\Big]$$
$$= \frac{1}{16}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(\ell-2)!}{(\ell+2)!}\int_0^\infty dk\, k\, |z_{2k\ell-}|^2\, |(a_{2k\ell m-}) - (a_{2,-k\ell m-})|^2 \cot(kT)\,. \quad (7.23)$$

From this expression, one proceeds as in [33,34] (for spin-0) and as above (for spin-1) to study the semi-classical quantum amplitude or wave function, proportional to $\exp\big(iS^{(2)}_{\rm class}\big)$, as a function of the complexified time-interval $T = |T|\exp(-i\theta)$, for $0 < \theta \leq \pi/2$. Just as in the spin-0 and spin-1 case, one straightforwardly recovers the complex Lorentzian amplitude for odd-parity gravitational perturbations, on taking the limit $\theta \to 0_+$.

8 Regge-Wheeler formalism – even-parity gravitational perturbations

Working with even-parity gravitational perturbations in the RW formalism is notoriously more difficult than working with those of odd parity. Yet, Chandrasekhar [57] showed that solutions to Zerilli's even-parity equation [64,70] [Eq.(9.10) below] can be expressed in terms of the odd-parity solutions. One might therefore expect that our results for the even-parity action should mirror those for the odd-parity action.

We expand the even-parity perturbations as

$$\begin{aligned}\Big(h^{(+)}_{ij}\Big)_{\ell m}(x) =&\, h^{(+)}_{1\ell m}(t\,,r)\Big[(f_1)_{ij}\Big]_{\ell m} + H_{2\ell m}(t\,,r)\,e^{(a-b)/2}\Big[(f_2)_{ij}\Big]_{\ell m} \\ &+ r^2 K_{\ell m}(t\,,r)\Big[(f_3)_{ij}\Big]_{\ell m} + r^2 G_{\ell m}(t\,,r)\Big[(f_4)_{ij}\Big]_{\ell m}\,,\end{aligned} \qquad (8.1)$$

Here, the non-zero components of the (un-normalised) basis of the symmetric tensor spherical harmonics $\big[(f_{1,2,3,4})_{ij}\big]_{\ell m}$ are defined by

$$\big[(f_1)_{r\theta}\big]_{\ell m} = \big(\partial_\theta Y_{\ell m}\big)\,, \qquad (8.2)$$

$$\big[(f_1)_{r\phi})\big]_{\ell m} \;=\; \big(\partial_\phi Y_{\ell m}\big) \quad , \tag{8.3}$$

$$\big[(f_2)_{rr}\big]_{\ell m} \;=\; Y_{\ell m} \quad , \tag{8.4}$$

$$\big[(f_3)_{\theta\theta}\big]_{\ell m} \;=\; Y_{\ell m} \quad , \tag{8.5}$$

$$\big[(f_3)_{\phi\phi}\big]_{\ell m} \;=\; \big(\sin^2\theta\big)\, Y_{\ell m} \quad , \tag{8.6}$$

$$\big[(f_4)_{\theta\theta}\big]_{\ell m} \;=\; \big(\partial_\theta\big)^2\, Y_{\ell m} \quad , \tag{8.7}$$

$$\big[(f_4)_{\theta\phi}\big]_{\ell m} \;=\; \big[\partial_\theta\partial_\phi - (\cot\theta)\partial_\phi\big] Y_{\ell m} \quad , \tag{8.8}$$

$$\big[(f_4)_{\phi\phi}\big]_{\ell m} \;=\; \big[(\partial_\phi)^2 + (\sin\theta\,\cos\theta)\partial_\theta\big] Y_{\ell m} \quad , \tag{8.9}$$

The non-zero inner products are given by

$$\int d\Omega\, \big[(f_1)^{ij}\big]_{\ell m}\, \big[(f_1)_{ij}\big]^*_{\ell' m'} \;=\; \frac{2e^{-a}}{r^2}\,\ell(\ell+1)\delta_{\ell\ell'}\,\delta_{mm'} \quad , \tag{8.10}$$

$$\int d\Omega\, \big[(f_2)^{ij}\big]_{\ell m}\, \big[(f_2)_{ij}\big]^*_{\ell' m'} \;=\; e^{-2a}\,\delta_{\ell\ell'}\,\delta_{mm'} \quad , \tag{8.11}$$

$$\int d\Omega\, \big[(f_3)^{ij}\big]_{\ell m}\, \big[(f_3)_{ij}\big]^*_{\ell' m'} \;=\; \frac{2}{r^4}\,\delta_{\ell\ell'}\,\delta_{mm'} \quad , \tag{8.12}$$

$$\int d\Omega\, \big[(f_4)^{ij}\big]_{\ell m}\, \big[(f_4)_{ij}\big]^*_{\ell' m'} \;=\; \frac{\Lambda_\ell(\Lambda_\ell-1)}{r^4}\,\delta_{\ell\ell'}\,\delta_{mm'} \quad , \tag{8.13}$$

$$\int d\Omega\, \big[(f_3)_{ij}\big]_{\ell m}\, \big[(f_4)^{ij}\big]^*_{\ell' m'} \;=\; -\,\frac{\ell(\ell+1)}{r^4}\,\delta_{\ell\ell'}\,\delta_{mm'} \quad , \tag{8.14}$$

where we define $\Lambda_\ell = \ell(\ell+1)$. The even-parity basis is also orthogonal to the odd-parity basis of Sec.4.

Further, for the even-parity perturbed shift, one can write

$$\Big[N_i^{(+)}\Big]_{\ell m} \;=\; \Big[H_{1\ell m}(t\,,r)\,Y_{\ell m}\,,\; h^{(+)}_{0\ell m}(t\,,r)\big(\partial_\theta Y_{\ell m}\big)\,,\; h^{(+)}_{0\ell m}(t\,,r)\big(\partial_\phi Y_{\ell m}\big)\Big]\,. \tag{8.15}$$

For the perturbed lapse,

$$\big[N^{(1)(+)}\big]_{\ell m} \;=\; -\,\frac{1}{2}\,H_{0\ell m}(t\,,r)\,e^{-a/2}\,Y_{\ell m} \quad . \tag{8.16}$$

Again, $H^*_{0\ell m} = (-1)^m\,H_{0\ell,-m}$, etc. Hence, for the linear-order perturbation $h^{(1)}_{\mu\nu}$ of the 4-dimensional metric, the quantities $h^{(1)}_{tt}$, $h^{(1)}_{rr}$ and $h^{(1)}_{tr}$ behave as scalars under rotations (their odd-parity part vanishes), while $h^{(1)}_{t\theta}$, $h^{(1)}_{t\phi}$, $h^{(1)}_{r\theta}$ and $h^{(1)}_{r\phi}$ behave as vectors. For $a\,,b = \theta,\phi$, the 2×2 angular block $h^{(1)}_{ab}$ is a tensor under rotations. The even-parity gravitational momentum components can, correspondingly, be written in the form

$$\begin{aligned}\big(\pi^{(+)}_{ij}\big)_{\ell m} = {}& ({}^3\gamma)^{\frac{1}{2}}\Big(P_{h_1\ell m}(t\,,r)\big[(f_1)_{ij}\big]_{\ell m} + P_{H_2\ell m}(t\,,r)\big[(f_2)_{ij}\big]_{\ell m} \\ & + r^2\,P_{K\ell m}(t\,,r)\big[(f_3)_{ij}\big]_{\ell m} + r^2\,P_{G\ell m}(t\,,r)\big[(f_4)_{ij}\big]_{\ell m}\Big)\,.\end{aligned} \tag{8.17}$$

Again, one can easily show that the P's in Eq.(8.17) are related to $h_1\,, H_2\,, K$ and G of the corresponding Eq.(8.1) by:

$$
\begin{aligned}
P_{h1\ell m}(t\,,r) &= \frac{1}{2}\,e^{a/2}\left(\left(\partial_t h^{(+)}_{1\ell m}\right) - r^2\,\partial_r\left(\frac{h^{(+)}_{0\ell m}}{r^2}\right)\right)\,, && (8.18)\\
P_{G\ell m}(t\,,r) &= \frac{1}{2}\,e^{a/2}\left(\left(\partial_t G_{\ell m}\right) - \left(\frac{2h^{(+)}_{0\ell m}}{r^2}\right)\right)\,, && (8.19)\\
P_{K\ell m}(t\,,r) &= -\,\frac{1}{2}\,e^{a/2}\Bigg(\left(\partial_t H_{2\ell m}\right) + \left(\partial_t K_{\ell m}\right) \\
&\quad + \left(a' - \frac{2}{r}\right)e^{-a}H_{1\ell m} - 2\,e^{-a}\left(\partial_r H_{1\ell m}\right) \\
&\quad + \left(\frac{2\ell(\ell+1)\,h^{(+)}_{0\ell m}}{r^2}\right) - \ell(\ell+1)\left(\partial_t G_{\ell m}\right)\Bigg)\,, && (8.20)\\
P_{H2\ell m}(t\,,r) &= -\,e^{a/2}\left(\partial_t K_{\ell m}\right) + \left(\frac{2}{r}\right)e^{-a/2}H_{1\ell m} - \left(\frac{\ell(\ell+1)\,h^{(+)}_{0\ell m}}{r^2}\right)e^{a/2} \\
&\quad + \frac{1}{2}\,\ell(\ell+1)e^{a/2}\left(\partial_t G_{\ell m}\right)\,. && (8.21)
\end{aligned}
$$

For even-parity gravitational perturbations, gauge transformations are induced by even-parity gauge vector fields $(\xi^{(+)\mu})_{\ell m}$, of the form:

$$
\begin{aligned}
&\left(\xi^{(+)t}\right)_{\ell m} = X^{(+)}_{0\ell m}(t\,,r)\,Y_{\ell m}\,, \quad \left(\xi^{(+)r}\right)_{\ell m} = X^{(+)}_{1\ell m}(t\,,r)\,Y_{\ell m}\,,\\
&\left(\xi^{(+)\theta}\right)_{\ell m} = X^{(+)}_{2\ell m}(t\,,r)\left(\partial_\theta Y_{\ell m}\right), \quad \left(\xi^{(+)\phi}\right)_{\ell m} = \left(\frac{X^{(+)}_{2\ell m}(t\,,r)}{\sin^2\theta}\right)\left(\partial_\phi Y_{\ell m}\right).
\end{aligned}
\qquad (8.22)
$$

Within the adiabatic approximation, these induce the following even-parity gauge transformations:

$$
\begin{aligned}
H'_{0\ell m} &= H_{0\ell m} - a' X^{(+)}_{1\ell m} + 2\left(\partial_t X^{(+)}_{0\ell m}\right)\,, && (8.23)\\
H'_{1\ell m} &= H_{1\ell m} - e^{-a}\left(\partial_r X^{(+)}_{0\ell m}\right) - e^{a}\left(\partial_t X^{(+)}_{1\ell m}\right)\,, && (8.24)\\
H'_{2\ell m} &= H_{2\ell m} - a'\,X^{(+)}_{1\ell m} - 2\left(\partial_r X^{(+)}_{1\ell m}\right)\,, && (8.25)\\
K'_{\ell m} &= K_{\ell m} - \left(\frac{2X^{(+)}_{1\ell m}}{r}\right)\,, && (8.26)\\
G'_{\ell m} &= G_{\ell m} - 2X^{(+)}_{2\ell m}\,, && (8.27)\\
h^{(e)'}_{0\ell m} &= h^{(+)}_{0\ell m} + e^{-a}X^{(+)}_{0\ell m} - r^2\left(\partial_t X^{(+)}_{2\ell m}\right)\,, && (8.28)\\
h^{(+)'}_{1\ell m} &= h^{(+)}_{1\ell m} - e^{a}X^{(+)}_{1\ell m} - r^2\left(\partial_r X^{(+)}_{2\ell m}\right)\,. && (8.29)
\end{aligned}
$$

As in the odd-parity case, we would like to construct gauge-invariant com-

binations of components of the perturbed 3-geometry. Following [68], we define

$$k_{1\ell m} \;=\; K_{\ell m} + e^{-a}\left(r\Big(\partial_r G_{\ell m}\Big) - \left(\frac{2h^{(+)}_{1\ell m}}{r}\right)\right) \quad , \tag{8.30}$$

$$k_{2\ell m} \;=\; \frac{1}{2}\left(e^{a}\, H_{2\ell m} - e^{a/2}\,\partial_r\Big(r\, e^{a/2}\, K_{\ell m}\Big)\right) \quad . \tag{8.31}$$

It can be shown that both the functions $k_{1\ell m}$ and $k_{2\ell m}$ are gauge-invariant. For future use, in the calculation of the even-parity classical action, we define [73] the linear combination of $k_{1\ell m}$ and $k_{2\ell m}$:

$$q_{1\ell m} \;=\; r\ell(\ell+1)k_{1\ell m} + 4re^{-2a}\, k_{2\ell m} \quad . \tag{8.32}$$

At this stage, as with the odd-parity case, we again make use of the property of the uniqueness of the (even-parity) RW gauge. In the RW gauge, one has

$$\begin{aligned} H^{RW}_{0\ell m} \;&= H_{0\ell m} - \frac{1}{2}\, r^2\, a'\, e^{-a}\left(\frac{2h^{(+)}_{1\ell m}}{r^2} - \Big(\partial_r G_{\ell m}\Big)\right) + r^2\, e^{a}\,\big(\partial_t\big)^2 G_{\ell m} \\ &\quad - 2e^{a}\Big(\partial_t h^{(+)}_{0\ell m}\Big) \quad , \end{aligned} \tag{8.33}$$

Then,

$$\begin{aligned} H^{RW}_{1\ell m} \;&=\; H_{1\ell m} + r^2\Big(\partial_r\partial_t G_{\ell m}\Big)\Big(\partial_r h^{(+)}_{0\ell m}\Big) - \Big(\partial_t h^{(+)}_{1\ell m}\Big) \\ &\quad + r\left(1 + \frac{1}{2}\, ra'\right)\Big(\partial_t G_{\ell m}\Big) - a'\, h^{(+)}_{0\ell m} \quad . \end{aligned} \tag{8.34}$$

Next,

$$\begin{aligned} H^{RW}_{2\ell m} \;&=\; H_{2\ell m} + \left(a' - \frac{4}{r}\right)e^{-a}\left(h^{(+)}_{1\ell m} - \frac{1}{2}r^2\Big(\partial_r G_{\ell m}\Big)\right) \\ &\quad + r^2\, e^{-a}\left(\big(\partial_r\big)^2 G_{\ell m} - 2\,\partial_r\left(\frac{h^{(+)}_{1\ell m}}{r^2}\right)\right) \quad . \end{aligned} \tag{8.35}$$

Further,

$$K^{RW}_{\ell m} \;=\; K_{\ell m} - \left(\frac{2e^{-a}}{r}\right)\left(h^{(+)}_{1\ell m} - \frac{1}{2}r^2\Big(\partial_r G_{\ell m}\Big)\right) \quad ; \tag{8.36}$$

with

$$G^{RW}_{\ell m} \;=\; 0 \;=\; G_{\ell m} - 2X^{(+)}_{2\ell m} \quad , \tag{8.37}$$

together with

$$h^{(+)RW}_{0\ell m} \;=\; 0 \quad , \tag{8.38}$$

and

$$h^{(+)RW}_{1\ell m} \;=\; 0 \quad , \tag{8.39}$$

where (in the RW gauge)

$$X_{0\ell m}^{(+)} = e^{-b}\left(\frac{1}{2}r^2\Big(\partial_t G_{\ell m}\Big) - h_{0\ell m}^{(+)}\right) \quad , \tag{8.40}$$

$$X_{1\ell m}^{(+)} = e^{-a}\left(h_{1\ell m}^{(+)} - \frac{1}{2}r^2\Big(\partial_r G_{\ell m}\Big)\right) \quad , \tag{8.41}$$

$$X_{2\ell m}^{(+)} = \frac{1}{2}G_{\ell m} \quad . \tag{8.42}$$

At late times, following gravitational collapse to a black hole, in the absence of background matter and in the adiabatic approximation, the even-parity RW field equations are seven coupled equations for the four unknowns $(H_{0\ell m}^{RW}, H_{1\ell m}^{RW}, H_{2\ell m}^{RW}, K_{\ell m}^{RW})$. Assuming that $\ell \geq 2$ — that is, that we are studying dynamical modes — we give here those RW field equations which are of first order in r and t [70]. These are, respectively, the $(t\theta)$, (tr) and $(r\theta)$ components of the linearised field equations:

$$\Big(\partial_r H_{1\ell m}^{RW}\Big) + \frac{2m}{r^2}e^a H_{1\ell m}^{RW} = e^a\,\partial_t\Big(K_{\ell m}^{RW} + H_{2\ell m}^{RW}\Big) \quad , \tag{8.43}$$

$$\begin{aligned}\frac{1}{2}\ell(\ell+1)H_{1\ell m}^{RW} =& -r^2\Big(\partial_t\partial_r K_{\ell m}^{RW}\Big) + r^2\Big(\partial_t H_{0\ell m}^{RW}\Big)\\ &- re^a\left(1-\frac{3m}{r}\right)\Big(\partial_t K_{\ell m}^{RW}\Big) \quad ,\end{aligned} \tag{8.44}$$

$$\Big(\partial_t H_{1\ell m}^{RW}\Big) = e^{-a}\Big(\partial_r H_{0\ell m}^{RW}\Big) - e^{-a}\Big(\partial_r K_{\ell m}^{RW}\Big) + \frac{2m}{r^2}H_{0\ell m}^{RW} \,, \tag{8.45}$$

and the $(\theta\phi)$ component

$$H_{0\ell m}^{RW} = H_{2\ell m}^{RW} \equiv H_{\ell m}^{RW} \quad . \tag{8.46}$$

We also give one second-order equation, namely, the (rr) component:

$$\begin{aligned}e^{2a}\big(\partial_t\big)^2 K_{\ell m}^{RW} =& \frac{2}{r}e^a\Big(\partial_t H_{1\ell m}^{RW}\Big) - \frac{1}{r}\Big(\partial_r H_{2\ell m}^{RW}\Big)\\ &+ \frac{e^a}{r}\left(1-\frac{m}{r}\right)\Big(\partial_r K_{\ell m}^{RW}\Big)\\ &- \frac{e^a}{2r^2}(\ell+2)(\ell-1)\Big(K_{\ell m}^{RW} - H_{2\ell m}^{RW}\Big) \quad .\end{aligned} \tag{8.47}$$

Following Eq.(8.44), we find, for the gauge-invariant component defined in Eq.(8.32):

$$\big(\partial_t q_{1\ell m}\big) = \ell(\ell+1)\left(r\Big(\partial_t K_{\ell m}^{RW}\Big) - e^{-a}H_{1\ell m}^{RW}\right) \quad . \tag{8.48}$$

We also find

$$\Big(\partial_r q_{1\ell m}\Big) = -\,\ell(\ell+1)\left(2e^{-a}k_{2\ell m} + \left(1+\frac{1}{2}ra'\right)k_{1\ell m}\right) \quad , \tag{8.49}$$

where, in the RW gauge,

$$q_{1\ell m} \quad \equiv \quad 2re^{-a}\, H^{RW}_{\ell m} - 2r^2\, e^{-a}\Big(\partial_r K^{RW}_{\ell m}\Big) + 2\big(\lambda r + 3m\big)K^{RW}_{\ell m} \quad , \tag{8.50}$$

with $\lambda = \frac{1}{2}(\ell+2)(\ell-1)$. We can now solve for $k_{1\ell m}$, $k_{2\ell m}$ in terms of $q_{1\ell m}$ and its radial derivative $(\partial_r q_{1\ell m})$, giving

$$k_{1\ell m} \;=\; \frac{1}{2\big(\lambda r + 3m\big)}\left(\left(\frac{re^{-a}\big(\partial_r q_{1\ell m}\big)}{(\lambda+1)}\right) + q_{1\ell m}\right) \quad , \tag{8.51}$$

$$k_{2\ell m} \;=\; -\,\frac{e^a}{4\big(\lambda r + 3m\big)}\left(r\big(\partial_r q_{1\ell m}\big) + e^a\left(1 - \frac{3m}{r}\right)q_{1\ell m}\right) . \tag{8.52}$$

Further, $H^{RW}_{\ell m}$ and $K^{RW}_{\ell m}$ can also be written in terms of $k_{1\ell m}$ and $k_{2\ell m}$.

9 Classical action and boundary conditions – even-parity gravitational perturbations

As in the case of odd-parity gravitational perturbations, we can exploit the uniqueness of the RW gauge to simplify the even-parity action and to obtain a general gauge-invariant form for the even-parity classical action $S^{(2)}_{\rm class}$. In the RW gauge, this is

$$\begin{aligned} S^{(2)}_{\rm class}\Big[(h^{(+)}_{ij})_{\ell m}\Big] \;=\; & \frac{1}{32\pi}\int_{\Sigma_F} d^3x \sum_{\ell\ell' mm'}\Big(\pi^{(+)ij}\Big)_{\ell m}\Big(h^{(+)}_{ij}\Big)^*_{\ell' m'} \\ & + \frac{1}{32\pi}\sum_{\ell m}\int_0^{R_\infty} dr\, e^{a/2}\Big(H^{RW*}_{\ell m}\, P^{RW}_{H2\ell m} + 2K^{RW*}_{\ell m}\, P^{RW}_{K\ell m}\Big)\Big|_T \end{aligned} . \tag{9.1}$$

Again, we would like to put the action into the form $\int dr\, \psi\, (\partial_t \psi)$, where ψ is gauge-invariant and obeys a decoupled wave equation. Since $q_{1\ell m}$ is the only unconstrained gauge-invariant even-parity quantity which involves only perturbations of the intrinsic 3-geometry, one might expect that Eq.(9.1) should reduce to the form

$$S^{(2)}_{\rm class}\Big[\{q_{1\ell m}\}\Big] \;=\; \frac{1}{32\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^{R_\infty} dr\Big(\pi_{1\ell m}\, q^*_{1\ell m} + \big(\partial_r Z_{\ell m}\big)\Big)\Big|_{t=T} \quad , \tag{9.2}$$

for some variable $Z_{\ell m}$, where $\pi_{1\ell m}$ is the gauge-invariant momentum conjugate to $q_{1\ell m}$. This is in fact the case. First, make in Eq.(9.1) the substitutions (as mentioned at the end of Sec.8) for $H^{RW}_{2\ell m}$ and $K^{RW}_{\ell m}$ in terms of $k_{1\ell m}$ and $k_{2\ell m}$; then substitute the expressions (8.51,52) for $k_{1\ell m}$ and $k_{2\ell m}$ in terms of $q_{1\ell m}$. After several integrations by parts, we arrive at an action of the form (9.2), with

$$\pi_{1\ell m} \;= \frac{r^2}{2\big(\lambda r + 3m\big)}\left(\hat{P}^{RW}_{\ell m} - \left(1 - \frac{3m}{r}\right)P^{RW}_{H_2\ell m}\, e^{3a/2}\right)$$

$$
- \frac{1}{2}\,\partial_r\biggl(\frac{r^3}{(\lambda r+3m)}\biggl(\frac{\hat{P}^{RW}_{\ell m}\,e^{-a}}{(\lambda+1)} - P^{RW}_{H_2\ell m}\,e^{a/2}\biggr)\biggr) \quad , \tag{9.3}
$$

$$
Z_{\ell m} \;=\; r^3\,e^{a/2}\,P^{RW}_{H_2\ell m}\,K^{RW}_{\ell m} + \frac{r^3\,q_{1\ell m}}{2(\lambda r+3m)}\biggl(\frac{\hat{P}^{RW}_{\ell m}\,e^{-a}}{(\lambda+1)} - P^{RW}_{H_2\ell m}\,e^{a/2}\biggr) \tag{9.4}
$$

$$
\hat{P}^{RW}_{\ell m} \;=\; e^{a/2}\Bigl(2P^{RW}_{K\ell m} - 2P^{RW}_{H_2\ell m} - r\bigl(\partial_r P^{RW}_{H_2\ell m}\bigr)\Bigr) \quad . \tag{9.5}
$$

This expression for the even-parity classical action simplifies yet further, since the linearised field equations imply that

$$
\hat{P}^{RW}_{\ell m} \;=\; \frac{(\lambda+1)}{r}\,e^a\,H^{RW}_{1\ell m} \quad . \tag{9.6}
$$

Further, Eqs.(8.43,44) show, with the help of Eq.(8.48), that

$$
\pi_{1\ell m} \;=\; \frac{\lambda r e^a}{2(\lambda r+3m)}\bigl(\partial_t q_{1\ell m}\bigr) \quad , \tag{9.7}
$$

Eq.(9.2) for the even-parity $S^{(2)}_{\rm class}$ then reduces to an expression of the desired form:

$$
\begin{aligned}
&S^{(2)}_{\rm class}\Bigl[\{\bigl(h^{(+)}_{ij}\bigr)_{\ell m}\}\Bigr] \\
&\quad = \frac{1}{32\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(\ell-2)!}{(\ell+2)!}\int_0^{R_\infty} dr\,e^a\,\xi^{(+)}_{2\ell m}\,\bigl(\partial_t\xi^{(+)*}_{2\ell m}\bigr)\Big|_{t=T} \; ,
\end{aligned} \tag{9.8}
$$

where $\xi^{(+)}_{2\ell m}$ is defined as

$$
\xi^{(+)}_{2\ell m} \;=\; \frac{\lambda r\,q_{1\ell m}}{(\lambda r+3m)} \quad . \tag{9.9}
$$

We have made use of the assumption above that the specified perturbations $h^{(1)}_{ij}|_{\Sigma_F}$ of the spatial 3-metric on the final boundary Σ_F have been taken to be real. Of course, for the Dirichlet boundary-value problem with T rotated into the complex, the classical solution for the metric and scalar field will have both an imaginary part and a real part.

Given the uniqueness of the RW gauge for even-parity modes, one can see that Eq.(9.8) for $S^{(2)}_{\rm class}$ is in fact valid in any gauge, with a vanishing contribution from the total divergence since $q_{1\ell m}$, as given by Eq.(8.28), and therefore also $\xi^{(+)}_{2\ell m}$, are gauge-invariant. There are obvious similarities between Eq.(9.8) and the classical massless-scalar action of [33,34], with $\xi^{(+)}_{2\ell m}$ and $\xi_{0\ell m+}$ differing only by an ℓ-dependent normalisation factor. This should not be surprising, as scalar spherical harmonics have even parity.

Again, one can show that the gauge-invariant quantity $\xi^{(+)}_{2\ell m}$ satifies Zerilli's equation [70]

$$
e^{-a}\,\partial_r\Bigl(e^{-a}\bigl(\partial_r\xi^{(+)}_{2\ell m}\bigr)\Bigr) - \bigl(\partial_t\bigr)^2\xi^{(+)}_{2\ell m} - V^{(+)}_\ell\,\xi^{(+)}_{2\ell m} \;=\; 0 \quad , \tag{9.10}
$$

$$V_\ell^{(+)} \;=\; \left(1-\frac{2m}{r}\right)\frac{2\Big(\lambda^2(\lambda+1)r^3+3\lambda^2mr^2+9\lambda m^2r+9m^3\Big)}{r^3\big(\lambda r+3m\big)^2} \quad > \quad 0\;. \tag{9.11}$$

Now, both for odd and even parity, the field equations for the metric perturbations have been reduced to the two wave equations (5.44) and (9.10).

In contrast to the odd-parity case, where we assumed an initially stationary multipole, here for even parity we treat $\xi^{(+)}_{2\ell m}$ by analogy with the massless-scalar-field quantity $\xi_{0\ell m+}$, and impose the Dirichlet boundary condition

$$\xi^{(+)}_{2\ell m}(0\,,r) \;=\; 0 \tag{9.12}$$

at the initial surface Σ_I $(t=0)$. Proceeding now by analogy with the separation-of-variables analysis of Sec.7 for the odd-parity case, we find that, if $K^{RW}_{\ell m}$ has $\sin(kt)$ time-dependence, then so must $H^{RW}_{\ell m}$ also, whereas $H^{RW}_{1\ell m}$ must have $\cos(kt)$ time-dependence. Consistency with the gauge transformations (8.33-39) implies that these time dependences are valid in an arbitrary gauge, and further that $G_{\ell m}$ and $h^{(+)}_{1\ell m}$ have $\sin(kt)$ time dependence, whereas $h^{(+)}_{0\ell m}$ has $\cos(kt)$ time dependence. Consequently, $q_{1\ell m}$ must have $\sin(kt)$ time dependence, whence the boundary condition (9.12) is justified through Eq.(9.9). (Alternatively, one could instead have studied normal-mode time dependence.)

Following the scalar-field analysis of [33,34], we can write

$$\xi^{(+)}_{2\ell m}(t\,,r) \;=\; \int_{-\infty}^{\infty} dk\, a^{(+)}_{2k\ell m}\,\xi^{(+)}_{2k\ell}(r)\,\frac{\sin(kt)}{\sin(kT)} \quad , \tag{9.13}$$

where the $\{a^{(+)}_{2k\ell m}\}$ are suitable even-parity 'Fourier coefficients', and where $\{\xi^{(+)}_{2k\ell m}(r)\}$ are real radial functions. These functions satisfy

$$e^{-a}\,\frac{d}{dr}\left(e^{-a}\,\frac{d\xi^{(+)}_{2k\ell}}{dr}\right)+\left(k^2-V^{(+)}_\ell(r)\right)\xi^{(+)}_{2k\ell} \quad = \quad 0 \quad . \tag{9.14}$$

Regularity at the origin implies that

$$\xi^{(+)}_{2k\ell}(r) \quad \sim \quad (\text{const.})\;\times\; r\,j_\ell(kr) \tag{9.15}$$

for small r. Again, at large r, the potential vanishes sufficiently rapidly that $\xi^{(+)}_{2k\ell}(r)$ has the asymptotic form

$$\xi^{(+)}_{2k\ell}(r) \quad \sim \quad \left(\left(z^{(+)}_{2k\ell}\right)\exp(ikr^*_s)+\left(z^{(+)*}_{2k\ell}\right)\exp(-ikr^*_s)\right) \quad , \tag{9.16}$$

where $\{z^{(+)}_{2k\ell m}\}$ are complex constants. Then, the classical action $S^{(2)}_{\text{class}}$ for even-parity gravitational perturbations reads

$$S^{(2)}_{\text{class}}\Big[\{a^{(+)}_{2k\ell m}\}\Big]$$
$$=\frac{1}{16}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(\ell-2)!}{(\ell+2)!}\int_0^\infty dk\,k\,|z_{2k\ell+}|^2\,|\big(a_{2k\ell m+}\big)+\big(a_{2,-k\ell m+}\big)|^2\cot(kT)\,, \tag{9.17}$$

where the notation is in line with that for spin-0 and for odd-parity fields. The coordinates $\{a_{2k\ell m+}\}$ label the configuration in k-space of the even-parity part of the metric perturbations on the final surface Σ_F .

Let us now re-assemble both the odd-and even-parity metric perturbations. As above, we consider for simplicity odd-parity metric perturbations which are initially static (Neumann problem) and even-parity metric perturbations which vanish initially (Dirichlet problem), on the space-like hypersurface Σ_I . The total classical spin-2 action is then

$$\begin{aligned} S^{(2)}_{\rm class} &= \frac{1}{32\pi}\sum_{\ell m P}\frac{(\ell-2)!}{(\ell+2)}\,P\int_0^{R\infty} dr\, e^{(a-b)/2}\,\xi_{2\ell mP}\Big(\partial_t\xi^*_{2\ell mP}\Big)\Big|_{\Sigma_F} \\ &= \frac{1}{16}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{P=\pm}\frac{(\ell-2)!}{(\ell+2)!}\int_0^{\infty} dk\, k|z_{2k\ell P}|^2|(a_{2k\ell mP})+(Pa_{2,-k\ell mP})|^2\cot(kT)\,, \end{aligned} \tag{9.18}$$

where the complex coefficients $\{a_{2k\ell mP}\}$ obey

$$a_{2k\ell mP} \quad = \quad P\,(-1)^m\,a^*_{2,-k\ell,-mP} \quad . \tag{9.19}$$

Here, P takes the value ± 1 according as the parity is even or odd.

As in the case of odd-parity metric perturbations (Sec.7), the even-parity metric perturbations also diverge at large r , except in a special gauge, the asymptotically-flat (AF) gauge. In the AF gauge for even parity, as for odd parity, all physical components $h^{(1)}_{(\mu)(\nu)} = |\gamma^{\mu\mu}\gamma^{\nu\nu}|^{\frac{1}{2}}\,h^{(1)}_{\mu\nu}$ (that is, all components of $h^{(1)}_{\mu\nu}$ projected onto the legs of a pseudo-orthonormal tetrad oriented along the unperturbed $(t\,,r\,,\theta\,,\phi)$ directions) fall off in the wave zone more rapidly than r^{-1} , except for the transverse (angular) components, which carry information about the gravitational radiation. In the new (AF) gauge, for even parity, one has

$$h^{(+)AF}_{0\ell m} \quad = \quad H^{AF}_{0\ell m} \quad = \quad H^{AF}_{1\ell m} \quad = \quad 0 \quad . \tag{9.20}$$

Further, from the even-parity gauge transformations (8.23-29), one has

$$0 \;=\; H^{RW}_{0\ell m} - a'\hat{X}^{(+)}_{1\ell m} + 2\Big(\partial_t\hat{X}^{(+)}_{0\ell m}\Big) \quad , \tag{9.21}$$

$$0 \;=\; H^{RW}_{1\ell m} + e^{-a}\Big(\partial_r\hat{X}^{(+)}_{0\ell m}\Big) - e^{a}\Big(\partial_t\hat{X}^{(+)}_{1\ell m}\Big) \quad , \tag{9.22}$$

$$H^{AF}_{2\ell m} \;=\; H^{RW}_{2\ell m} - a'\,\hat{X}^{(+)}_{1\ell m} - 2\Big(\partial_r\hat{X}^{(+)}_{1\ell m}\Big) \quad , \tag{9.23}$$

$$K^{AF}_{\ell m} \;=\; K^{RW}_{\ell m} - \left(\frac{2\hat{X}^{(+)}_{1\ell m}}{r}\right) \quad , \tag{9.24}$$

$$G^{AF}_{\ell m} \;=\; -\,2\hat{X}^{(+)}_{2\ell m} \quad , \tag{9.25}$$

$$0 \;=\; e^{-a}\,\hat{X}^{(+)}_{0\ell m} - r^2\Big(\partial_t\hat{X}^{(+)}_{2\ell m}\Big) \quad , \tag{9.26}$$

$$h^{(+)AF}_{1\ell m} \;=\; -\,e^{a}\hat{X}^{(+)}_{1\ell m} - r^2\Big(\partial_r\hat{X}^{(+)}_{2\ell m}\Big) \quad , \tag{9.27}$$

where a hat denotes a gauge function in the AF gauge. Therefore, once given $H^{RW}_{\ell m}$, $H^{RW}_{1\ell m}$ and $K^{RW}_{\ell m}$, then Eqs.(9.21,22) can be solved for $\hat{X}^{(+)}_{0\ell m}$ and $\hat{X}^{(+)}_{1\ell m}$. Thence, Eq.(9.26) can be used, in order to solve for $\hat{X}^{(+)}_{2\ell m}$. In solving these equations, one chooses the arbitrary functions which arise such that asymptotic flatness is still satisfied. Thus, the AF gauge is consistent.

10 Conclusion

In this paper, we have taken over the scalar (spin-0) calculations of [33,34], with the help of the angular harmonics of Regge and Wheeler [63], to include the more complicated Maxwell (spin-1) and linearised graviton (spin-2) cases. For spin-1, the linearised Maxwell field splits into a part with even parity and a part with odd parity; a different treatment is needed for each of these two cases. In both cases the relevant boundary conditions involve fixing the magnetic field on the initial space-like boundary Σ_I and final boundary Σ_F . The main result is an explicit expression (6.19) for the classical (linearised) Maxwell action, as a functional of the final magnetic field, subject to the simplifying assumption that the magnetic field on the initial surface Σ_I is zero. From this, the Lorentzian quantum amplitude for photon final data can be derived, as in [34] for spin-0 perturbative final data, by taking the limit $\theta \to 0_+$ of $\exp\bigl(iS_{\rm class}\bigr)$, where $S_{\rm class}$ is the action of the classical solution of the boundary-value problem with prescribed initial and final data, with complexified time-interval $T = |T| \exp(-i\theta)$, where $0 < \theta \leq \pi/2$.

Linearised gravitational-wave ($s = 2$) perturbations about a spherically-symmetric Einstein/massless-scalar collapse to a black hole have also been studied here. As for Maxwell ($s = 1$) perturbations, the principal aims for $s = 2$ also are (1) to specify suitable perturbative boundary data on the final space-like hypersurface Σ_F at a late time T , subject (for simplicity) to the initial boundary data on Σ_I (time $t = 0$) being spherically symmetric; (2) to express the spin-2 Lorentzian classical action $S_{\rm class}$ as an explicit functional of the 'suitable' boundary data above, and of the proper-time interval T , once T has been rotated into the complex: $T \to |T| \exp(-i\theta)$, for $0 < \theta \leq \pi/2$; (3) given $S_{\rm class}$, to compute, following Feynman, the quantum amplitude for the weak-field final data, by taking the limit of the semi-classical amplitude (const.) $\times \exp\bigl(iS_{\rm class}\bigr)$ as $\theta \to 0_+$.

As in the $s = 1$ case, it is also necessary for $s = 2$ to decompose the metric perturbations into parts with odd and even parity. The main difference on moving from the $s = 1$ to the $s = 2$ case is a considerable increase in algebraic or analytic complexity, to be expected since one deals with tensor fields rather than vector fields.

Some indications towards unification of these ideas for perturbative fields of different spin s appear already in Secs. 2,3. For $s = 1$, the quantity most naturally specified as an argument of the quantum wave-functional, on a bounding hypersurface such as Σ_F , is the (spatial) magnetic field B_i , subject to the condition ${}^3\nabla_k B^k = 0$. Correspondingly, for linearised gravitational waves

($s = 2$), the natural boundary data were found to be the (symmetric, trace-free) magnetic part H_{ik} of the Weyl tensor [52,53], subject to ${}^3\nabla_k H^{ik} = 0$. In 2-component spinor language, these correspond ($s = 1$) to a particular 'projection' of the (complex) symmetric Maxwell field-strength spinor $\phi_{AB} = \phi_{(AB)}$, and ($s = 2$) to a corresponding projection of the totally-symmetric (complex) Weyl spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$. Of course, as treated in [35], these boundary conditions constructed from ϕ_{AB} and Ψ_{ABCD} are special cases of the natural boundary conditions for gauged supergravity [71-73]. In our work on quantum amplitudes for spin-$\frac{1}{2}$ [35], the natural boundary conditions also involved a corresponding projection of the spin-$\frac{1}{2}$ field. Although 2-component spinor language might (to some people) seem a luxury in treating bosonic fields describing photons or gravitons, above, it is practically a necessity in treating the corresponding fermionic (massless) neutrino spin-$\frac{1}{2}$ field, as in [35], and (for supergravity) the gravitino spin-$\frac{3}{2}$ field, on which work is in progress [36].

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